Advances in Mathematics **155**, 98–153 (2000)
doi:10.1006/aima.2000.1932, available online at http://www.idealibrary.com on IDEAL®

Some New Results on Eigenvectors via Dimension, Diameter, and Ricci Curvature

Dominique Bakry

CNRS and Université Paul Sabatier, 118 Route de Narbonne, 31062 Toulouse, France

and

Zhongmin Qian

Department of Mathematics, Imperial College of Science, Technology, and Medicine, 180 Queen's Gate, London SW7 2BZ, United Kingdom; and

Received July 22, 1999; accepted April 12, 2000

We generalise for a general symmetric elliptic operator the different notions of dimension, diameter, and Ricci curvature, which coincide with the usual notions in the case of the Laplace–Beltrami operators on Riemannian manifolds. If λ_1 denotes the spectral gap, that is the first nonzero eigenvalue, we investigate in this paper the best lower bound on λ_1 one can obtain under an upper bound on the dimension, an upper bound on the diameter, and a lower bound of the Ricci curvature. Two cases are known: namely if the Ricci curvature is bounded below by a constant $R>0$, then $\lambda_1 \geqslant nR/(n-1)$, and this estimate is sharp for the n-dimensional spheres (Lichnerowicz's bound). If the Ricci curvature is bounded below by zero, then Zhong–Yang's estimate asserts that $\lambda_1 \geqslant \frac{\pi^2}{d^2}$, where d is an upper bound on the diameter. This estimate is sharp for the 1-dimensional torus. In the general case, many interesting estimates have been obtained. This paper provides a general optimal comparison result for λ_1 which unifies and sharpens Lichnerowicz and Zhong–Yang's estimates, together with other comparison results concerning the range of the associated eigenfunctions and their derivatives.

1. INTRODUCTION

The determination of lower bounds for spectral gaps (or equivalently Poincaré inequalities) for (sub) elliptic symmetric operators is an important issue, in many domains of mathematics from statistical mechanics to differential geometry, since this constant determines the rate of convergence to equilibrium in dissipative evolution systems. Moreover, it is also quite

0001-8708/00 $35.00

important to get some information on the eigenvectors, such as their range or their image measure through the Riemann measure.

Many results have been obtained in the case of the Laplace–Beltrami operators of Riemannian manifolds (we shall call them Laplacians), under various hypotheses on curvature, diameter, and dimension, and a large number of papers on spectral geometry have been written by various authors over the past years. It is out of the author's capacity to trace all these excellent works, we only cite several monographs like [6, 7, 11, 12, 14, 15, 27, 43] for further references. Among them, many works have been dedicated to obtaining optimal bounds under lower bounds on Ricci curvature and upper bounds on the diameter.

The main purpose of this paper is to provide several new comparison theorems about eigenfunctions and eigenvalues via dimension, diameter and Ricci curvature. Those notions may be generalised to general elliptic operators, and the results obtained in this paper are optimal in this respect. In particular, the notion of dimension we introduce here coincides with the usual notion of dimension only in the case of Laplacians, and it is not an integer in general.

Many of the results presented here are extensions of previous work by Kröger in the case of Laplacians [34], and the authors thank gratefully P. Li for having driven their attention to Kröger's paper which was very close in spirit to a first version of this present paper, even though the technical details of the proofs are quite different. More precisely, Kröger's work makes an important use of the linearity of the problem, while our approach could be extended (with some extra work which is not done here) to some nonlinear setting like the determination of optimal bounds on Sobolev or Logarithmic Sobolev constants.

But even when dealing with Laplacians, it is sometimes important to consider more general elliptic operators, and the reader will see some examples of this situation below.

Anyway, our results apply in particular to Riemannian manifolds and we therefore give comparison theorems for closed (and Neumann) eigenvalue and eigenvectors via lower bounds of the Ricci curvature.

Let us describe the results in the Riemannian case for a start. If (M, g) is a compact Riemannian manifold of dimension n with or without boundary, let Δ be its Laplace–Beltrami operator. The spectral gap $\lambda_1(M)$ of the manifold (M, g) is defined to be the first nonzero eigenvalue of the Laplacian Δ: the smallest positive constant λ such that there is a nonconstant function u satisfying that

$$\Delta u = -\lambda u, \qquad \text{on} \quad M; \qquad \frac{\partial u}{\partial \nu} = 0, \qquad \text{on } \partial M,$$

where v denotes the normal direction on the boundary, when it is nonempty.

Cheeger [16] and Cheng [23] (also see [3, 29, 32]) have proved that the spectral gap $\lambda_1(M)$ is bounded above by a quantity depending only on the dimension, the diameter, and a lower bound of the Ricci curvature. In fact, for the Dirichlet eigenvalue problem, Cheng [23] obtained an upper bound via a lower bound of Ricci curvature and a lower bound via an upper bound of sectional curvature.

Also, by the fundamental works of Lichnerowicz [38], Li [35], and Li and Yau [36, 37], the lower bound of the spectral gap $\lambda_1(M)$ can also be estimated via the dimension, the diameter, and a lower bound of Ricci curvature. See [6, 7, 13, 14, 16, 24, 28, 30, 37, 43, 49], only a few mentioned, see also the references therein.

Among all these estimates, two results are optimal: Lichnerowicz's theorem and Zhong-Yang's result: in those two cases, the authors compare the spectral gap under lower bounds of Ricci with the spectral gap of some Riemannian manifold satisfying the same conditions (the space form).

More precisely, if $\mathrm{Ric} \geqslant (n-1)K$, $K \geqslant 0$, then the famous Lichnerowicz's estimate [38] for the spectral gap asserts that $\lambda_1(M) \geqslant nK$. Later Obata [41] proved that equality holds if and only if the manifold M is isometric to the n-dimensional sphere with constant curvature K. However if $\mathrm{Ric} \geqslant 0$, i.e. $K=0$, then Lichnerowicz's estimate is useless.

On the other hand, in 1979, Li [35] proved that if $\mathrm{Ric} \geqslant 0$, then $\lambda_1(M) \geqslant \frac{\pi^2}{2d^2}$, where d is the diameter of the manifold M. Li's estimate has been improved to the sharp Zhong–Yang's estimate [52] $\lambda_1(M) \geqslant \frac{\pi^2}{d^2}$, which is optimal since the lower bound is achieved on the one-dimensional torus.

We may rewrite these two results as comparison theorems within the same framework. Lichnerowicz's estimate asserts the following: if M is an n-dimensional Riemannian manifold with Ricci curvature bounded below by $(n-1)K > 0$, then $\lambda_1(M) \geqslant \lambda_1(n, K)$, where $\lambda_1(n, K)$ is the spectral gap of the n-dimensional sphere with Ricci curvature $(n-1)K$. Therefore it is a comparison theorem for spectral gaps via dimension and lower bound on the Ricci curvature, the model space here being the sphere.

Since the first nonconstant eigenvectors on spheres are radial, the Lichnerowicz lower bounds are in fact a comparison result with the radial part of the Laplace–Beltrami operator on spheres. Assume $K=1$ for simplicity: the Laplace–Beltrami operator preserves radial functions, and, acting on them, becomes the Jacobi operator

$$L = \frac{d^2}{dr^2} + (n-1)\cot(r)\frac{d}{dr}$$

on the interval $[0, \pi]$. Here, π appears as the diameter of the sphere. For reasons which will be explained later on, we choose to move this interval to the centred one $[-\pi/2, \pi/2]$, so that the operator becomes

$$L = \frac{d^2}{dr^2} - (n-1)\tan(r)\,\frac{d}{dr}.$$

Now we can state Lichnerowicz's estimate as

THEOREM 1 [Lichnerowicz]. *Let M be a compact Riemannian manifold of dimension n (with a convex boundary), and let* $\mathrm{Ric} \geqslant (n-1)$. *Then*

$$\lambda_1(M) \geqslant \lambda_1(n-1, n, \pi),$$

where $\lambda_1(n-1, n, \pi)$ is the first nonzero eigenvalue of the Neumann problem:

$$v'' - (n-1)\tan v' = -\lambda v, \qquad \text{on} \quad \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], v'\left(-\frac{\pi}{2}\right) = v'\left(\frac{\pi}{2}\right) = 0.$$

Of course, in this context, this eigenvalue is n, and the function $\sin(x)$ is the associated eigenvector. Also, the diameter π appears naturally in this estimate since, under those hypotheses, the diameter is bounded above by π by Myers' theorem.

We may restate the Li and Zhong–Yang's estimate along the same line. This time the Ricci curvature is bounded below by zero, and therefore the comparison object should be the radial Laplacian of the space form $\mathbb{R}^n$ (with curvature zero) on some interval of length d: the diameter of the manifold. More precisely, the operator is the Jacobi operator

$$L = \frac{d^2}{dr^2} + \frac{n-1}{r}\,\frac{d}{dr}.$$

It remains to determine which is the good interval with which to compare. As in Theorem 1, the answer given by Zhong–Yang's estimate is that the good choice is the symmetric interval with respect to the *centre* and length d. However the two end points are zero and $+\infty$, and therefore the centre point is also $+\infty$. Since the length of the interval is finite, the operator on this interval becomes $L = d^2/(dr^2)$ as $r \to \infty$, and the dimension disappears.

Notice that $L = d^2/(dr^2)$ is invariant under translations, and therefore we can choose any interval with length d. Now we are in a position to restate Li and Zhong–Yang's result as a comparison theorem.

THEOREM 2 [Peter Li [35], Zhong–Yang [52]]. *Let M be a Riemannian manifold of dimension n (with a convex boundary) and diameter d, and let* $\mathrm{Ric} \geqslant 0$. *Then*

$$\lambda_1(M) \geqslant \lambda(0, \infty, d),$$

where $\lambda(0, \infty, d)$ is the first nonzero eigenvalue of the Neumann problem:

$$v'' = -\lambda v, \qquad \text{on} \quad \left(-\frac{d}{2}, \frac{d}{2}\right), \qquad v'\left(-\frac{d}{2}\right) = v'\left(\frac{d}{2}\right) = 0.$$

We shall explain in the next chapter the notions of dimension, diameter and curvature for general operators. By anticipation, let us say that the previous Lichnerowicz's estimate appears as a comparison theorem with a Jacobi operator of same "dimension" n, constant "Ricci curvature" $(n-1)K$ and diameter π, and Li and Zhong–Yang's estimate is a comparison theorem with a Jacobi operator with dimension n, zero Ricci curvature and diameter d.

One of the aspects of the results of this paper is that it unifies those two results into a general comparison theorem under lower bounds on the Ricci curvature, upper bounds on the diameter, and upper bounds on the dimension. It again appears as a general comparison theorem, but in the general case, we compare the spectral gap with the spectral gap of an operator satisfying the same conditions and which is not a Laplacian. The notions of Ricci curvature, dimension, and diameter for general operators that we introduce in the next chapter will coincide with the usual Riemannian notions in the case of Laplacians.

To be short, we shall concentrate a lower bound R on the Ricci tensor and upper bound n on the dimension into an inequality $CD(R, n)$ (see Eq. 5 of the next section). Then, we shall give a complete answer to the question of the best lower bounds of spectral gaps under the condition $CD(R, n)$, given an upper bound d on the diameter, which is valid even when the dimension n is infinite. The answer is as follows: the best lower bound than one can get is the (Neumann) eigenvalue of a one dimensional operator satisfying the inequality $CD(R, n)$ on the interval $[-d/2, d/2]$. The explicit form of this operator is described below (there are 6 different basic models, according to the fact that the lower bound R on the Ricci curvature is positive or negative and to the fact that the dimension n is finite or not).

Given an operator satisfying a curvature-dimension inequality $CD(R, n)$, we shall compare different objects related to this operator (gradient of eigenvectors, maxima of eigenvectors, repartition of invariant measures around the extrema of eigenvectors, etc.) to the corresponding objects related to a one-dimensional model $L_{R,n}$ described as follows:

(1) For real constants $R>0$ and $n>1$, $L_{R,n}$ is the operator defined on

$$\left(-\pi/2\sqrt{\frac{R}{n-1}},\ \pi/2\sqrt{\frac{R}{n-1}}\right)$$

by

$$L_{R,n}(v)(x)=v''(x)-\sqrt{R(n-1)}\,\tan\left(\sqrt{\frac{R}{n-1}}\,x\right)v'(x);$$

(2) For $R<0$ and $n>1$, $L_{R,n}$ is the operator on an extended real line

$$(0,\infty)\cup(-\infty,\infty)\cup(-\infty,0)=I_1\cup I_2\cup I_3 \text{ by}$$

$$L_{R,n}(v)(x)=v''(x)+\sqrt{-(n-1)R}\,\coth\left(\sqrt{\frac{-R}{n-1}}\,x\right)v'(x) \qquad \text{on} \quad I_1\cup I_3.$$

$$L_{R,n}(v)(x)=v''(x)+\sqrt{-(n-1)\,R}\,\tanh\sqrt{\frac{-R}{n-1}}\Bigg)\,v'(x) \qquad \text{on } I_2\,.$$

(3) For $n>1$ and $R=0$ the operator $L_{0,n}$ is defined on $I_1\cup I_2\cup I_3$ by

$$L_{0,n}(v)(x)=v''(x)+\frac{n-1}{x}\,v'(x), \qquad \text{on} \quad I_1\cup I_3\,,$$

and

$$L_{0,n}(v)(x)=v''(x) \qquad \text{on} \quad I_2.$$

We may have suppressed central interval I_2 and glued together the infinity point of I_1 and I_3: this makes no difference in the final result.

(4) Finally, for $n=\infty$, $R\neq 0$, the operators $L_{R,\infty}$ are the operators defined on the real line by

$$L_{R,\infty}(v)(x)=v''(x)-Rxv'(x),$$

while the special case $L_{0,\infty}$ consists of the full family of operators on the real line

$$L_{0,\infty}(v)(x)=v''(x)-av'(x),$$

where a is a constant.

We will return to the model operators $L_{R,n}$ at the end of Sections 2 and 3, where the reader may find a more detailed description of these families. The reader will see there that they all come from the solutions of a simple differential equation related to the curvature-dimension inequality. Observe that all those one-dimensional operators may be written as $v''-T(x)\,v'$ on some interval: we call this function $T(x)$ the "drift term" of $L_{R,n}$.

For the moment, as a consequence of the comparison theorems on eigenvectors given in this paper, we get the following results for Riemannian manifolds, which were obtained in Kröger's paper [34] by similar methods (see Section 8 for the complete set of results):

THEOREM 3. *Let M be a compact Riemannian manifold of dimension n (with a convex boundary or without boundary), let d be an upper bound of it's diameter, and assume that* $\mathrm{Ric}\geqslant(n-1)\,K=R$. *We know that $d\leqslant\pi/\sqrt{K}$ if $K>0$. Then $\lambda_1(M)\geqslant\lambda_1(R,n,d)$, where $\lambda_1(R,n,d)$ denotes the first non-zero eigenvalue of the Neumann problem*:

$$L_{R,n}(v)=-\lambda v,\qquad \text{on}\quad \left(-\frac{d}{2},\frac{d}{2}\right),\qquad v'\left(-\frac{d}{2}\right)=v'\left(\frac{d}{2}\right)=0.$$

In this theorem, in the negative curvature case, the interval $(-d/2,d/2)$ is considered as a subinterval of I_2, which means that we consider the operator with the function $\tanh(x)$.

Although the results look very similar, the situation of the positive and negative bounds on the curvature are in fact rather different. Indeed, in the positive case, the operator $L_{R,n}$ is the radial part of the Laplace–Beltrami operator of the space form (the n-dimensional sphere), while the operator $L_{R,n}$ on I_2 is not the radial part of the hyperbolic space in the negative case, which would be $L_{R,n}$ on I_1.

If we follow the idea of the flat case given by the Zhong-Yang's result, we would consider this operator at the interval centred at infinity, which means that we would consider the operator

$$L(v)=v''+\sqrt{-R(n-1)}\,v',$$

on an interval of length d. The Neumann spectral gap for this operator is easy to compute and always larger than $((n-1)\,R)/4$: it is easy to check that this cannot be a lower bound for λ_1 under our hypotheses. In fact we shall prove that for any subinterval in I_1 (with the function $\coth(x)$), the Neumann spectral gap for $L_{R,n}$ on this interval is too big to be a lower bound.

Indeed, we must go beyond the infinity point to reach the central point: $L_{R,n}$ on I_2 is the prolongation of our radial Laplacian beyond infinity: this operator in this context is no longer the radial part of the hyperbolic Laplacian, but still related to the Laplace–Beltrami operator of the hyperbolic space and is still an operator of constant negative curvature in the sense of the next chapter.

We may write the operator as $L_{R,n} = v'' - T(x)\, v'$ and consider the topology on $I_1 \cup I_2 \cup I_3$ which makes the drift term $T(x)$ continuous. It means that we must glue the $+\infty$ of I_1 with the $+\infty$ of I_2, and the $-\infty$ of I_2 with the $-\infty$ of I_3, and think $I = I_1 \cup I_2 \cup I_3$ as a big interval. Since we shall always consider those operators on subintervals of finite length, there will be no problem to determine on which part we are looking at it. But we will move continuously those subintervals from the boundary one $(0, a) \subset I_1$ for example, to the central one $(-a/2, a/2)$ on I_2, such that the quantities relative to those operators (spectral gaps, maxima of eigenvectors, etc.) move continuously from the boundary situation to the central one.

We would like to point out that Theorem 3 sharpens both Lichnerowicz's and Li and Zhong–Yang's lower bounds on the spectral gaps. Indeed, under the condition that

$$\mathrm{Ric} \geqslant R, \qquad d \leqslant \frac{\pi}{\sqrt{R/(n-1)}}$$

by Myers' finite diameter theorem [3, 40] and therefore

$$\lambda_1(R, n, d) \geqslant \lambda_1\left(R, n, \pi \middle/ \sqrt{\frac{R}{n-1}}\right) = nR/(n-1).$$

On the other hand, by choosing $R = 0$, we get the Zhong–Yang's estimate.

Our proof makes no fundamental distinction between the positive and negative cases. Let us say a few words about it before going into the details.

Basically, the proofs of Lichnerowicz and Li and Zhong–Yang results possess different nature. The first proof comes easily from an integration by parts argument and makes full use of the invariant measure of the Laplace–Beltrami operator, while the second proof relies on a precise upper bound of the gradient of an eigenvector. It is therefore not surprising that our result relies on a mixture of both arguments. Let us describe the main steps in the case of Laplacians.

To prove Theorem 3, we will first prove Kröger's comparison on the gradient of eigenvectors, which is the key tool in these studies. We state it in the Riemannian setting for the moment.

THEOREM 4. (Kröger, [34]). *Let M be a compact Riemannian manifold of dimension n, with Ricci curvature bounded below by a constant R. Let u be an eigenfunction with a nonzero eigenvalue λ, and let v be a solution on some interval (a, b) of the differential equation*

$$L_{R,n}(v) = -\lambda v, \qquad \text{on } (a, b), \qquad v'(a) = v'(b) = 0, \tag{1}$$

such that $v' \neq 0$ on (a, b) and $[\min u, \max u] \subset [\min v, \max v]$. Then

$$|\nabla(v^{-1} \circ u)| \leqslant 1. \tag{2}$$

Observe that this result is indeed a comparison theorem: let u be an eigenvector with eigenvalue λ on our manifold. Consider an interval (a, b) which has λ as first nonzero eigenvalue for the Neumann problem on our model operator $L_{R,n}$, v being the corresponding eigenvector. Then if the range of u is included in the range of v, at any points x and y such that $u(x) = v(y)$, the gradient of u at x is bounded above by the gradient of v at y.

Theorem 4 can be applied to the central solution of the differential equation (4). That is, if v is an eigenfunction with eigenvalue λ_1 of the Neumann problem

$$L_{R,n}v = -\lambda_1 v, \qquad \text{on } (-l, l);\ v'(-l) = v'(l) = 0,$$

then we can renormalize v and u so that they fit the conditions that $\min v = v(-l) = \min u$, and $\max u \leqslant -\min u$. Since v is odd, we have $\min v = -\max v$, so that $\max u \leqslant \max v$. Therefore for this v, we have $|\nabla v^{-1} \circ u| \leqslant 1$.

To deduce the proof of Theorem 3 when $\max u = -\min u$ is then straightforward, since then we may choose for v the eigenvector of the one-dimensional model for the central interval, which satisfies the same relationship by symmetry. See Corollary 6 of Section 5.

This does not work when $\min v = \min u$ but $\max u < \max v$. Actually, this is the main difficulty of the whole problem, and, in the case of the zero Ricci curvature, it is the main difference between Li's result [35] and Zhong–Yang's result. If we can find an interval $I = (a, b)$ for which $\max u = \max v$ (and $\min u = \min v$), then Theorem 4 and the above argument still yield that the diameter $d \geqslant b - a$. Thus, to prove Theorem 3, we shall prove that there always exists an interval (a, b) among all intervals which has the same eigenvalue than u for the one-dimensional model operator, and such that the corresponding eigenvector v satisfies $\max u = \max v$ and $\min u = \min v$. We shall deduce this from a comparison result on the ratio of minima and maxima of eigenvectors, which relies on an integration by parts argument and requires a lower bound of volumes of small balls. This requirement will always be satisfied for Riemannian manifolds and for elliptic operators with smooth coefficients. This result will allow us to compare the eigenvalue of the manifold to one of the eigenvalues of the one-dimensional models with the same diameter (which means the eigenvalue of the corresponding one-dimensional model on some subinterval of length d).

In the end, it remains to prove that among all the intervals having the same eigenvalue, the central interval possesses the smallest length, or equivalently that among all the intervals of a given length, the central interval has the lowest eigenvalue. This result itself will be deduced from the above gradient comparison theorem, but this time applied to one-dimensional operators instead of Laplacians. This is achieved in Section 7.

The first point is achieved in Section 6, by the following comparison theorem for the maximum values (when the minimums are fixed) for the ground states.

Let us denote B_R the "left" boundary point of the maximal interval on which $L_{R,n}$ is defined, that is, the point where the drift term $T(x)$ goes to

$$-\infty\colon\ B_R = \frac{-\pi}{\sqrt{R/(n-1)}}$$

if $R > 0$ and B_R is the 0 point of I_1 when $R \leqslant 0$.

Theorem 5. *Let M be a compact manifold of dimension n (without boundary or with a convex one), and suppose that* $\mathrm{Ric} \geqslant R$. *Let u be an eigenfunction with a nonzero eigenvalue λ_1, satisfying that* $\min u = -1$. *Let v be the solution of the differential equation*

$$L_{R,n} v = -\lambda_1 v, \qquad v'(B_R) = 0, \qquad v(B_R) = -1.$$

Define $\max v = v(x_1)$, *where x_1 is the first zero point of v' after B_R. Then,* $\max u \geqslant \max v$.

The reverse inequality is true for the "right" point, where the drift term goes to $+\infty$.

In the end, we put all the arguments together in Section 8 to give the main theorems of the paper. The reader interested in the scheme of the proof could go directly to this section and see that the proof follows in the end from the fundamental ideas of Li.

Let us mention that it is not necessary to get those estimates on the maxima of eigenvectors if one is just concerned with the lower bound on eigenvalues. Kröger overcomes this difficulty by comparing the maximum of the eigenvactor with the maximum value of an (boundary) eigenvector with a bigger dimension, and concludes by an comparison argument on the diameter of one dimensional boundary eigenvectors.

This lower bound on maxima itself is obtained through a comparison result on the ratios

$$\frac{\int_{u \leqslant c} u\mu(dx)}{\int_{u \leqslant d} u\mu(dx)}$$

with the same quantity for the eigenvector of the model operator, where $c < d < 0$ and μ is the invariant measure for the operator (the Riemannian measure in the case of Laplacians) (see 12 below).

Notations. Throughout this paper, the letter T will denote a real function on an interval which satisfies $T' = R + T^2/(n-1)$. We will be able to restrict ourselves to the cases $R = n-1, 0, -(n-1)$, when n is finite, and $R = 1, 0, -1$ for infinite n, although any other value is allowed. When $R = n-1$, we will take T to be the function $(n-1)\tan$, and if $R = -(n-1)$, T will be $-(n-1)\tanh$ or $-(n-1)\coth$, except otherwise specified, and the operator $L_{R,n}$ is always $v'' - Tv'$.

2. CURVATURE-DIMENSION INEQUALITY

In the study of eigenvalues for Riemannian manifolds, lower bounds of Ricci curvature often appear through a curvature-dimension inequality which makes sense in the general context of diffusion generators. This is what we describe below. Although the notions we introduce make sense for any generator of a diffusion on a measurable space (and most of the result presented in this paper apply in this context), we shall restrict our attention to generators of diffusions on smooth manifolds.

Let M be a smooth connected p-dimensional manifold, and consider a second order (sub-) elliptic differential operator L on it: to fix the ideas, let us write L in a local chart as

$$L(u) = \sum_{i,j} a^{ij}(x) \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} u + \sum_i \frac{\partial}{\partial x^i} u,$$

where (a^{ij}) is a symmetric non negative matrix. All coefficients are assumed to be smooth.

There is no need to assume that (a^{ij}) is nondegenerate (the elliptic case), but it will be in fact the case of interest. When this happens, then the inverse matrix (a_{ij}) defines a Riemannian metric g on M, and the operator L may be written as

$$L = \Delta + B,$$

where Δ is the Laplace–Beltrami operator associated with the metric g and B is a vector field (that is, a first order differential operator).

All the notions we shall use to analyse lower bounds on the spectrum may completely be described from the operator L itself, without further reference to the differential structure of the space M. To start with, the scalar product (for the Riemannian metric) of the gradients ∇f and ∇g of two smooth functions may be defined as

$$\Gamma(f, g) = \frac{1}{2}\left[L(fg) - fLg - gLf\right] = \sum_{ij} a^{ij} \frac{\partial}{\partial x^i} f \frac{\partial}{\partial x^j} g,$$

and we may see then that the vector field X of the previous decomposition plays no rôle there.

The diameter of M associated with this operator L is defined by

$$d := \sup_{\{f : \Gamma(f, f) \leqslant 1\}} \{f(x) - f(y)\}.$$

In the elliptic case, this coincides with the usual Riemannian diameter associated with the metric g.

We may then define the operator Γ_2 by iterating this construction, which leads us to the definition

$$\Gamma_2(f, g) = \tfrac{1}{2}\left[L\Gamma(f, g) - \Gamma(Lf, g) - \Gamma(f, Lg)\right]. \tag{3}$$

This will be the object of interest in what follows. In the Riemannian case, that is when $L = \Delta$, then the Bochner identity [8] says

$$\Gamma_2(f, f) = |\text{Hess}\, f|^2 + \text{Ric}(\nabla f, \nabla f), \tag{4}$$

where Ric denotes the Ricci tensor and Hess$f$ is the Hessian tensor of the function f, the norm being the Hilbert-Schmidt norm in the Riemannian metric. Therefore, in this case, a lower bound R on the Ricci curvature is entirely characterised by the fact that, for any smooth function f, one has

$$\Gamma_2(f, f) \geqslant R\Gamma(f, f).$$

Moreover, if p denotes dimension of the space, then, since in this case $L(f) = \Delta(f)$ is the trace of Hess(f), we have

$$|\mathrm{Hess}\, f|^2 \geqslant \frac{1}{p}\,(Lf)^2,$$

and therefore, for any smooth function f,

$$\Gamma_2(f,f) \geqslant \frac{1}{p}\,(Lf)^2 + R\Gamma(f,f). \tag{5}$$

In what follows, we shall refer to this inequality as the curvature dimension inequality $CD(R,p)$. Moreover, it is also clear that, if the inequality $CD(R,n)$ holds for any smooth function f, then $n \geqslant p$ and R is a lower bound of the Ricci tensor.

From now on, this inequality will serve as definitions for curvature and dimension of a general differential operator L: more precisely, given any operator L as described before, and two reals R and $n \in (1, \infty]$, we shall say that its Ricci curvature is bounded below by R and that it's dimension is bounded above by n if the inequality $CD(R,n)$ holds for any smooth function f.

We do not exclude here the case $n = \infty$: it just means that we forget about the dimension in the estimates we are looking for (dimension free estimations). Moreover, there are operators of fundamental importance, like Ornstein-Uhlenbeck operators, which satisfy the $CD(1,\infty)$ inequality but no finite dimensional inequality.

Notice here that, from what preceeds, the notion of Ricci curvature for such operators comes before the notion of dimension: one may have operators with Ricci bounded below and infinite dimension, while one cannot consider any upper bound on dimension if the Ricci curvature is not bounded below.

We just recall the definition of an upper bound on diameter associated with L: it's diameter is bounded above by d if, for any smooth function f with $\Gamma(f,f) \leqslant 1$, then, for any two points (x, y) in the M, $f(x) - f(y) \leqslant d$.

Of course, everything is made as to coincide with the usual Riemannian notions in the case of Laplacians.

Let us describe what does this curvature dimension inequality means in the general elliptic case where $L = \Delta + B$, Δ being here the Laplace–Beltrami operator associated with some Riemannian structure.

Let ∇^s_B be the symmetric covariant derivative of B, namely,

$$\nabla^s_B(\xi, \eta) = \tfrac{1}{2}[\langle \nabla_\xi B, \eta \rangle + \langle \nabla_\eta B, \xi \rangle], \qquad \forall \xi, \eta \in TM.$$

Then, if Ric denotes the Ricci tensor of the Riemannian manifold, let us denote by $\mathrm{Ric}(L)$ the symmetric tensor

$$\mathrm{Ric}(L) = \mathrm{Ric} - \nabla^s_B.$$

In this case,

$$\Gamma_2(f, f) = |\mathrm{Hess}\, f|^2 + \mathrm{Ric}(L)(\nabla f, \nabla f),$$

and the $CD(R, n)$ inequality holds if and only if

$$n \geqslant p \text{ and } \mathrm{Ric}(L) - \frac{1}{n-p} B \otimes B \geqslant R Id,$$

as symmetric tensors.

So we may see that the dimension n is not in general equal to the geometric dimension p, unless the operator L is a Laplacian. Moreover, there are in general many different best choices for n and R in the $CD(R, n)$ inequalities (but there are operators which have a unique best choice for both n and R and which are not Laplacians).

It is certainly also interesting to observe that the change of L into cL for some positive constant c changes d into $d/\sqrt{c}$, preserves the dimension and changes R into cR. This explains why we may always reduce to one of the cases $R = (n-1)$, $R = -(n-1)$, and $R = 0$, when n is finite.

As particular examples of interest are the one-dimensional operators on the real line, or on an interval: up to a change of coordinate, such an operator may always be written under the form

$$L = \frac{d^2}{dx^2} - T(x) \frac{d}{dx}.$$

For this operator,

$$\Gamma(f, f) = f'^2; \qquad \Gamma_2(f, f) = f''^2 + T' f'^2,$$

and this operator satisfies the $CD(R, n)$ inequality if and only if $n \geqslant 1$ and the function $T(x)$ satisfies the differential inequality

$$T' \geqslant R + \frac{T^2}{n-1}. \tag{6}$$

We may set $R=(n-1)\,K$ and $T=(n-1)\,t$, such that this inequality becomes $t' \geqslant K+t^2$.

Obviously, the extremal choices for such operators are obtained when the above inequality becomes an identity, in which case our operators are the operators $L_{R,n}$ considered in the Introduction.

Observe that, in this last example, the Riemannian distance is the usual distance on the real line, and therefore that for those operators, the diameter is the length of the interval on which we consider them.

Observe also that, in the particular case of the $CD(1, \infty)$ inequality, the inequality becomes $T' \geqslant 1$, and that the extremal case is then the Orenstein-Uhlenbeck operator

$$L=\frac{d^2}{dx^2}-x\,\frac{d}{dx}.$$

The operators we are working with are second-order differential operators; they therefore satisfy a simple change of variable formula that one may write as follows: for any smooth real function f on M, and any smooth function φ from $\mathbb{R}$ into itself, then

$$L(\varphi(f))=\varphi'(f)\,Lf+\varphi''(f)\,\Gamma(f,f).$$

It is easy to deduce from that the change of variable formulas for $\Gamma(.,.)$ and $\Gamma_2(.,.)$.

One of the basic feature of the $CD(R,n)$ inequality is that it reinforces itself by a simple application of those chain rules. We have

THEOREM 6. *Let L be a second-order differential operator. Assume that, for some $n \in (1, \infty]$ and $R \in \mathbb{R}$, L satisfies the $CD(R,n)$ inequality. Then for any function f,*

$$\Gamma_2 \geqslant \frac{1}{n}\,(Lf)^2+R\Gamma+\frac{n}{n-1}\left[\frac{Lf}{n}-\frac{\Gamma(f,\Gamma)}{2\Gamma}\right]^2, \qquad on \quad \Gamma \neq 0, \qquad (7)$$

where $\Gamma=\Gamma(f,f)$ and $\Gamma_2=\Gamma_2(f,f)$.

We shall call this inequality the extended $CD(R,n)$ inequality. By the remark which follows the proof of the theorem, we shall see that it is sharp.

Proof. For simplicity, let

$$A(f,f)=\Gamma_2(f,f)-\frac{1}{n}\,(Lf)^2-R\Gamma(f,f).$$

Then, for any smooth function φ, we have

$$\begin{aligned} A(\varphi(f), \varphi(f)) = \varphi'(f)^2\, A(f, f) \\ + \varphi'\varphi'' \left[\Gamma(f, \Gamma) - \frac{2}{n}\, Lf\Gamma(f, f) \right] \\ + \varphi''^2 \left(1 - \frac{1}{n}\right) \Gamma(f, f)^2. \end{aligned}$$

Our hypothesis is that this quantity is always non negative; if we let φ vary among of C^2 functions, we get a quadratic form in the coefficients φ', φ'' which is positive, and therefore its discriminant is nonnegative, that is,

$$4A(f, f)\, \frac{n-1}{n}\, \Gamma^2(f, f) \geqslant \left[\Gamma(f, \Gamma) - \frac{2}{n}\, Lf\Gamma \right]^2,$$

which yields Inequality 7.

Remark. When L is one of our extremal operators $L_{R, n}$, then, the inequality 7 becomes an identity. Indeed, if

$$L = \frac{d^2}{dr^2} - T\, \frac{d}{dr}, \qquad \text{with} \quad T' = R + T^2/(n-1),$$

then

$$\Gamma_2(f, f) - R\Gamma(f, f) = (f'')^2 + \frac{T^2}{n-1}\, (f')^2.$$

On the other hand,

$$\begin{aligned} &\frac{1}{n}\, (Lf)^2 + \frac{n}{n-1} \left[\frac{Lf}{n} - \frac{\Gamma(f, \Gamma)}{2\Gamma} \right]^2 \\ &= \frac{1}{n}\, [f''^2 - 2Tf'f'' + T^2(f')^2] \\ &\quad + \frac{n}{n-1} \left[\frac{f''}{n} - \frac{T}{n}\, f' - f'' \right]^2 \\ &= f''^2 + T^2(f')^2. \end{aligned} \tag{15}$$

Therefore in this case,

$$\Gamma_2 = R\Gamma + \frac{1}{n}\, (Lf)^2 + \frac{n}{n-1} \left[\frac{Lf}{n} - \frac{\Gamma(f, \Gamma)}{2\Gamma} \right]^2.$$

The inequality is still true for $CD(R, \infty)$ inequality, and we have in this case

THEOREM 7. *Let L be a second-order differential operator on a smooth manifold, which satisfies the $CD(R, \infty)$ inequality, for some constant R. Then*

$$\Gamma_2 \geqslant R\Gamma + \left[\frac{\Gamma(f, \Gamma)}{2\Gamma}\right]^2, \qquad \text{on} \quad \Gamma \neq 0,$$

where $\Gamma_2 = \Gamma_2(f, f)$ and $\Gamma = \Gamma(f, f)$.

Of course, we could have proved directly this extended $CD(R, n)$ inequality in the case of Laplacians by considering a local system of coordinates which diagonalises at some point x the Ricci tensor. The same proof in the case of a general elliptic operator becomes rapidly very tedious. The advantage of working with intrinsic objects like the Γ and Γ_2 operators is that they give directly the sharpest results.

At last, at the end of this chapter, let us describe more precisely the different one-dimensional models we are going to work with. As we already mentioned, we may only consider the operators $L_{\pm(n-1), n}$, $L_{0, n}$, $L_{\pm 1, \infty}$, $L_{0, \infty}$. All the other operators are obtained by multiplying the previous by some positive constant c^2, and then changing x into x/c. We thus obtain a full two parameter family of operators, which is continuous in the naïve obvious topology.

First, the one-dimensional model for $CD(n-1, n)$ is defined on the symmetric interval $(-\pi/2, \pi/2)$ by

$$L_{n-1, n} = \frac{d^2}{dx^2} - (n-1)\tan(x)\frac{d}{dx}.$$

It is preserved by the symmetry $x \leftrightarrow -x$, which leaves invariant the central point 0 where $T(x) = 0$. Its reversible measure is the measure

$$\mu_{n-1, n}(dx) = \cos(x)^{n-1}\, dx.$$

Observe that the drift $T(x) = (n-1)\tan(x)$ is singular at the boundaries $\pm\pi/2$, and that the measure of the boundary balls $B(\pm\pi/2, r) = [-\pi/2, -\pi/2 + r]$ or $[\pi/2 - r, \pi/2]$ with respect to this measure behaves like cr^n when r goes to 0; the measure of the other balls behave linearly with r when r goes to 0.

The one-dimensional operator model for $CD(-(n-1), n)$ is slightly more complicated: the space on which it acts is still symmetric: it is the union $I_1 \cup I_2 \cup I_3$, where $I_1 = (0, \infty)$, $I_2 = (-\infty, \infty)$, $I_3 = (\infty, 0)$. The operator is

$$L_{-(n-1), n} = \frac{d^2}{dx^2} + (n-1) \tanh(x) \frac{d}{dx} \qquad \text{on } I_2$$

and

$$L_{-(n-1), n} = \frac{d^2}{dx^2} + (n-1) \coth(x) \frac{d}{dx} \qquad \text{on } \quad I_1 \cup I_3 .$$

It is also preserved by the symmetry $x \leftrightarrow -x$ (which exchanges I_1 and I_3) and it's invariant measure is

$$\mu_{-(n-1), n}(dx) = \cosh(x)^{n-1}\, dx \qquad \text{on } I_2,$$

and

$$\mu_{-(n-1), n}(dx) = |\mathrm{sh}(x)|^{n-1}\, dx \qquad \text{on } \quad I_1 \cup I_3.$$

Observe that we could have written this operator on a finite interval by a simple change of coordinates which maps infinite intervals on finite ones (using the arc tan function for example), but then the natural distance would not have been preserved for our intervals.

As we already mentioned, it is natural to endow this space $I_1 \cup I_2 \cup I_3$ with the natural topology for which the drift term $T(x)$ is continuous. Then, there are two boundary points where the drift is singular (the 0 points of I_1 and I_3, and, as before, the measure of the balls $B(x, r)$ centred at the boundary points behaves like cr^n when r goes to 0.

There is also a central point: the 0 point of I_2, where $T(x) = 0$, and the symmetry $x \to -x$, which exchanges I_1 and I_3 and preserves I_2 leaves the operator invariant.

Similarly, the one dimensional operator model space for $CD(0, n)$ is the union $I_1 \cup I_2 \cup I_3$, and the operator is

$$L_{0, n} = \frac{d^2}{dx^2} + \frac{n-1}{x} \frac{d}{dx} \qquad \text{on } \quad I_1 \cup I_3,$$

and

$$L_{0, n} = \frac{d^2}{dx^2} \qquad \text{on } I_2$$

The invariant measure is

$$\mu_{0,n}(dx) = |x|^{n-1}\,dx$$

on $I_1 \cup I_3$ and is the Lebesgue measure on I_2. The drift $T(x)$ continuous on $I_1 \cup I_2 \cup I_3$ and the boundary points are those where $T(x)$ goes to infinity. Once again, the measure of the small balls centred at the boundaries and radius r behave like cr^n. The central point where $T(x)=0$ is the 0 point of I_2, and the symmetry $x \to -x$, which exchanges I_1 and I_3, leaves the operator invariant.

The one-dimensional models for $CD(1,\infty)$ and $CD(-1,\infty)$ are the operators defined on the real line by

$$L_{\pm 1,\infty} = \frac{d^2}{dx^2} - \pm x\,\frac{d}{dx},$$

again preserved by symmetry and with invariant measure

$$\mu_{\pm 1,\infty}(dx) = \exp(-\pm x^2/2)\,dx.$$

The central point where $T(x)=0$ is still the fixed point of the symmetry. But this time the boundary points where $T(x) \to \infty$ are at infinity, and the measure of small balls at infinity no longer behaves like cr^n (this reflects the fact that we are working here with infinite dimension).

Finally, the model space for $L_{0,\infty}$ is more peculiar: it is the family of all the operators

$$L^a_{0,\infty} = \frac{d^2}{dx^2} + a\,\frac{d}{dx}$$

acting on $\mathbb{R}$. The invariant measures are $\exp(-ax)\,dx$ and the symmetry $x \to -x$ exchanges $L^a_{0,\infty}$ with $L^{-a}_{0,\infty}$. The central operator corresponds is $L^0_{0,\infty}$ and the boundary points correspond to the limits $a \to \pm\infty$.

We shall not use this last family of operators, since every result concerning it may be obtained first by considering $L_{R,\infty}$ with some $R<0$, and then go to the limit when $R \to 0$.

To summarize, we may observe that, for any value of the curvature and dimension, we have a one dimensional model acting on an extended interval; they all share the following properties: there is a central point where the drift is 0, a symmetry around the central point under which the operator is invariant, and two boundary points where the drift is infinite.

3. SOME PROPERTIES OF THE ONE DIMENSIONAL MODELS

We collect in this chapter several elementary (and less elementary) properties of the eigenvectors and eigenvalues associated with our one-dimensional models that we need in the proofs of our main results. Most of them are completely elementary, and we shall just point out the most delicate points.

In all what follows, L will denote one of the one-dimensional model operators described before. Since there are many different models (depending if R is positive, negative or 0 and if the dimension n is finite or not), we shall just mention arguments which work in all the cases. When a property needs more careful inspection depending on the choice of the operator, then we shall mention it.

Remind that, in any case, the operators under considerations are defined on various intervals of $\mathbb{R}$ and have the form

$$L(v) = v'' - T(x)\, v',$$

where T is the solution of the following differential equation

$$T' = R + \frac{T^2}{n-1},$$

with different values of the parameters $R \in \mathbb{R}$, $n \in (1, \infty]$, and that we may always choose $R = (n-1)$, $R = -(n-1)$ or $R = 0$ in the finite dimension case, and $R = 1, 0, -1$ in the infinite dimensional case.

The choice of T is made such that, if the interval contains 0, then $T(0) = 0$. Also, it is always an odd function.

The function ρ will denote a solution of $T = -\rho'/\rho$, which is explicit in all the cases under consideration, and defined up to a constant which is unimportant in what follows. The function ρ is the density with respect to the Lebesgue measure of the reversible measure of our operator.

3.1. *Eigenvectors and Eigenvalues for the Neumann Problem*

We first give some elementary properties of solutions of one-dimensional differential equations, which will be used later in order to prove our main results. For a detailed content, the reader may refer for example to [42, 46].

If $I = (a, b)$ is a subinterval of $\mathbb{R}$, then we will use $\lambda_1(I)$ to denote the first nonzero eigenvalue of the Neumann problem on I:

$$L(v) = -\lambda v, \qquad \text{on } (a, b), \qquad v'(a) = v'(b) = 0. \tag{8}$$

This eigenvector has the property that $v' \neq 0$ on (a, b), and this condition determines the value of λ_1. We may always require an initial condition $v(a) = -1$, to fix the ideas. Then the function v is entirely determined by those initial conditions.

It is a general property of one-dimensional equations that if $I_1 \subset I_2$, then $\lambda_1(I_1) \geqslant \lambda_1(I_2)$, and that equality holds if and only if $I_1 = I_2$. This is not the case for Neumann boundary conditions in general, but it is here an easy consequence of the fact that the equation satisfied by v may be rewritten as $(\rho v')' = -\lambda_1 \rho v$, and an integration by parts consideration.

It is then easily seen that $\lambda_1 \geqslant \lambda_0 := \max\{0, nR/(n-1)\}$. The previous inequality also is strict if the interval is not equal to the maximum interval. That is, $(-\pi/2, \pi/2)$ in the case of $L_{n-1,n}$ or any bounded interval in all the other cases.

We first begin with symmetric bounded intervals $I = (-l, l)$. (In the particular case of $L_{0,n}$, this is simply the operator d^2/dx^2, which appears to be also $L_{0,\infty}$, on this interval.)

In this case, by the uniqueness of the solution, the eigenvector v is odd, and therefore we have $\max v = -\min v$. The function which maps l to $\lambda_1([-l, l])$ is monotone increasing, and it is simpler to consider it's inverse function $l(\lambda)$, which may defined as follows: $l(\lambda)$ is the first 0 of v', where v is the solution of the differential equation

$$L(v) = -\lambda v; \qquad v(0) = 0; \qquad v'(0) = 1.$$

It is important to notice that, for any $\lambda > \lambda_0$, this first 0 point is attained before the boundary of the interval, that is before $\pi/2$ in the case of $L_{1,n}$ for finite n when $\lambda > n$, and at finite distance in all the other cases. To see that, we must separate the cases of $L_{n-1,n}$ and $L_{1,\infty}$ (the positive curvature cases) from the others.

Indeed, the function $r = \lambda \frac{u}{u'}$ satisfies the differential equation

$$r' = \lambda - Tr + r^2.$$

In the nonpositive curvature cases, the function T is nonpositive on $\{x \geqslant 0\}$ ($T = -\tanh(x)$ if $R = -(n-1)$, $n > 1$, and $T(x) = -x$ if $R = -1$, $n = -\infty$), and therefore

$$r' \geqslant \lambda + r^2.$$

This shows that r goes to infinity at finite distance, which means exactly that l is finite.

In the other cases ($R > 0$, n finite or infinite), it is slightly more delicate and comes from considerations concerning the comparison between the eigenvector v and the eigenvector v_0 associated with the eigenvalue λ_0. The usual techniques mentioned above should be used with caution there, because in one case the drift term $T(x)$ is singular at the boundary, and in the other case, the interval is not bounded. But it is still an elementary exercise to check that everything works.

Moreover, it is standard to observe that the function $d(\lambda)$ is decreasing, continuous, goes to 0 when λ goes to infinity, and goes to infinity when λ goes to λ_0 in every case, except the finite dimension strictly positive curvature case where it goes to $\pi/2$ for $L_{n-1,n}$.

3.2. *Continuity of Maxima*

The next step is to observe what happens when looking at nonsymmetric intervals. In this section, we fix $\lambda > \lambda_0$, and denote $d = d(\lambda)$. We consider now a point $a < -d$, and the solution v_a of the equation

$$L(v_a) = -\lambda v, \qquad v'(a) = 0, \qquad v(a) = -1.$$

It is clear that $v_a''(a) > 0$, and therefore that $v_a' > 0$ in a right neighbourhood of a. If we call $b(a)$ the first zero of v_a' after a: the previous considerations show that $b(a)$ is finite, and even that $b(a) < d$. Then, $\lambda = \lambda_1([a, b(a)])$, and v_a is the corresponding eigenvector.

We then define the quantity $m(a)$ to be maximum value of v_a over the interval $[a, b(a)]$, which means that $m(a) = v_a(b(a))$. We are concerned in this section with the range of the function $m(a)$.

Let us remark first that, since v_a is an eigenvector for the Neumann problem, then

$$\int_a^{b(a)} v_a(t)\, \rho(t)\, dt = 0,$$

and therefore that $m(a) > 0$.

It should be observed too that since our operators are invariant under the change of x into $-x$, we could as well have looked at the eigenvectors on $a > -d$ up to the point where $b(a)$ reaches the boundary (when such happens). We have

$$v_{-b(a)}(x) = -\frac{v_a(-x)}{m(a)},$$

and therefore

$$m(-b(a)) = \frac{1}{m(a)}.$$

To analyse the range of $m(a)$, and although our results are identical in all the situations, we must distinguish between the many cases of curvature and dimension. We shall only deal with the operators $L_{n-1,n}$ and $L_{-(n-1),n}$, and the reader will convince himself that the results are exactly identical for the remaining particular cases.

Let us start with the positive curvature case, where $L_{n-1,n}v = v'' - (n-1)\tan(x)\,v'$. By the standard variation properties of ordinary differential equations, the function $a \to m(a)$ is continuous, as long as one does not reach the boundary of the intervals, where the drift coefficient $T(x)$ is singular.

Also, although the drift $T(x)$ is singular at the boundary point $-\pi/2$, the existence and uniqueness of a solution of the equation $L(v) = -\lambda v$ on $[-\pi/2, \pi/2)$ with boundary value $v(-\pi/2) = -1$, $v'(-\pi/2) = 0$ is easily obtained by standard fixed point techniques, using the fact that the equation $Lv = -\lambda v$ is in fact equivalent to

$$(\rho v')' = -\lambda \rho v,$$

where $\rho = -T'/T$ is the density of the invariant measure.

It is quite easy to observe that the second derivative of $v''_a(a)$ is not continuous at the boundary $-\pi/2$.

Proposition 1. *Let $\lambda > n$, $d = d(\lambda)$, and $a \in [-\pi/2, -d]$. Let v_a be the solution of the differential equation*

$$L_{n-1,n}(v) = -\lambda v, \qquad v'(a) = 0, \quad v(a) = -1.$$

We denote $v_{-\pi/2}$ by v. Then

$$\lim_{a \to -\pi/2} m(a) = m(-\pi/2). \tag{9}$$

Proof. We will prove that, for any $T < \pi/2$,

$$\lim_{a \to -\pi/2} v_a(x) = v(x), \qquad \lim_{a \to -\pi/2} v'_a(x) = v'(x)$$

for any $x \in (-\pi/2, T]$, which will yield the conclusion. Denote by $V = v - v_a$ the difference and by g the module $|V|$. In this context, the function ρ is $\rho(x) = \cos(x)^{n-1}$. We know that the differential equation satisfied by v_a can be rewritten

$$(\rho v_a')' = -\lambda \rho v_a, \tag{10}$$

and therefore

$$(\rho V')' = -\lambda \rho V. \tag{11}$$

Integrating (11) over the interval $[a, x]$, we obtain that

$$\int_a^x (\rho V')' = -\lambda \int_a^x \rho V(r)\, dr. \tag{12}$$

Using the fact that

$$\rho(a)\, V'(a) = \rho(a)\, v'(a),$$

we get from (12) that

$$V'(x) = v'(a)\, \frac{\rho(a)}{\rho(x)} - \lambda \int_a^x \frac{\rho(r)}{\rho(x)}\, V(r)\, dr. \tag{13}$$

Integrating again (13) over $(a, x]$, we get

$$V(x) = 1 + v(a) + v'(a)\, \rho(a) \int_a^b \frac{1}{\rho(r)}\, dr \tag{14}$$

$$-\lambda \int_a^x \left[\int_y^x \frac{\rho(y)}{\rho(r)}\, dr \right] V(y)\, dy. \tag{15}$$

Note that

$$\left| \int_y^x \frac{\rho(y)}{\rho(r)}\, dr \right|$$

is bounded above by a constant β depending only on n and T, Since $1 + v(a)$ goes to 0 when a goes to $-\pi/2$, then the above majorization shows the result.

As a corollary, and if we use the symmetry argument described previously, we obtain the following:

COROLLARY 1. *Let* $m = m(-\pi/2)$. *For any real* $m_1 \in [m, 1/m]$, *there exists a subinterval of* $[-\pi/2, \pi/2]$ *which has Neumann eigenvalue* λ *for* $L_{1,n}$, *and a corresponding eigenvector* v *such that* $\min(v) = -1$, *max*$(v) = m_1$.

Remark. As a consequence of the main gradient comparison theorem of this paper, one may show effectively that $m < 1$, and that the function $m(a)$ is indeed increasing (see Corollary 5 in Section 5). But we do not need this result here, and if we do not know that $m < 1$, one may change m into $1/m$. (See Corollary 5 below.)

The case of negative curvature is slightly more complicated. Let us just describe what happens. If we start with the operator $L_{-(n-1),n}$ on I_2, where it is

$$L_{-(n-1),n} v = v'' + (n-1) \tanh v',$$

it might be true that the length of the interval $[a, b(a)]$ goes to infinity when $a \to -\infty$: this happens when $4\lambda \leqslant (n-1)^2$. This is easy to observe since this value is the critical value for the operator in the limit, which is $v'' - (n-1)\, v'$. Then, for those values of λ, we do observe that $m(a)$ goes to infinity when $a \to -\infty$. For those values, we do not need more since the range of $m(a)$ covers the interval $(1, \infty)$, and by the symmetry, all the range $(0, \infty)$.

If λ is too big, then the length goes to a finite limit, and the function v_a itself goes to a limit which corresponds to the solution of the equation

$$v'' - (n-1)\, v' = -\lambda v,$$

with the usual boundary conditions. Then, on the interval $(-\infty, 0)$, the solution of the equation

$$v'' + (n-1) \coth(x)\, v' = -\lambda v$$

starting at a with the same boundary conditions goes to the same limit when a goes to $-\infty$.

The same argument proves that, when a goes to 0, the eigenvalue v_a and its derivative v'_a go to the boundary values v_0 and v'_0. (We just have to change $\cos(x)^{n-1}$ into $\mathrm{sh}(x)^{n-1}$.)

The function $m(a)$ is decreasing in this context, and to derive our conclusions, it seems more natural to change x into $-x$, using the previous symmetry argument and consider the interval $(0, \infty)$ instead of $(-\infty, 0)$.

COROLLARY 2. *Suppose $\lambda > 0$, and let $m = m(0)$ be the maximum value of the solution of the equation*

$$v'' + (n-1)\coth(x)\, v' = -\lambda v,$$

with initial conditions $v(0) = -1$, $v'(0) = 0$ (m may be finite or infinite). Then, for any $m_1 \in [m, 1/m]$, there exists an interval which has Neumann eigenvalue λ for $L_{-(n-1), n}$ and a corresponding eigenvector v with $\min(v) = -1$ *and* $\max(v) = m_1$.

Remark. In fact, we could see that this value m is 0 as soon as $\lambda < (n-1)^2/4$, but we were unable to prove that the distance $b(a) - a$ is an decreasing function of a on $(0, \infty)$, which we strongly conjecture.

Also, it may be that the value m_1 corresponds to the maximum of the eigenvector when a goes to infinity, in which case the associated operator $L_{-(n-1), n}$ is

$$v'' \pm (n-1)\, v',$$

on any real interval.

In the infinite dimensional cases $L_{1, \infty}$ and $L_{-1, \infty}$, it can be shown that the maximum value $m(a)$ goes to infinity when a goes to $+\infty$ and to 0 when a goes to $-\infty$ in the positive curvature case, and the reverse is true in the negative curvature case. Therefore, the full range of the maximum function is $(0, \infty)$. The same is true for the family of $L_{0, \infty}$ operators.

4. A DEEPER INSIGHT INTO ONE-DIMENSIONAL MODELS

In the previous section, we looked at some properties of the eigenvectors of the one-dimensional models. Those results were not really specific of the model, and we could have obtained roughly the same consequences for any one-dimensional models having the same boundary behaviour. In this section, we shall really use the fact that our models are given by

$$L(v) = v'' - T(x)\, v',$$

where $T' = R + T^2/(n-1)$.

Here, the parameter $n > 1$ may be infinite.

This section is the hardest part of the paper, and we prove here the main technical points which will be needed in the next sections.

Throughout this section, we consider an finite interval (a, b) on which the function T is a bounded solution of the ordinary differential equation

$$T' = R + \frac{T^2}{n-1}.$$

In the case where n is finite, we may assume that $R = \pm(n-1)$, and, if we set $T = (n-1)\,t$, then, up to a translation, we may always assume that on our interval t is the function $\tan(x)$, $\tanh(x)$ or $\coth(x)$, or $t = \pm 1$. We do not consider the latter case, which is simpler to handle and the reader will easily check that all the assertions of this chapter remain valid in this last case, where everything may be explicitly computed.

Observe therefore that T' is always strictly monotonic on (a, b) (except in the limiting case that we excluded, where it is constant).

In this section, we fix some $\lambda > \max\{0, (nR)/(n-1)\}$, and consider the solution v of the differential equation

$$v'' = T(x)\,v' - \lambda v,$$

with boundary conditions $v(a) = -1$, $v'(a) = 0$. The relation between b and λ is such that $v'(b) = 0$, and $v' > 0$ on (a, b). Notice that $v(b) > 0$. Throughout this section, x_0 will denote the unique point of (a, b) where $v(x_0) = 0$.

PROPOSITION 2. *If X denotes the function $\frac{\lambda v}{v'}$, then*

$$X' > \max\left\{0, \frac{n}{n-1}\,T'\right\} \qquad \text{on } (a, b).$$

Proof. Observe first that from our hypotheses, the function X goes to $\pm\infty$ at the boundaries of the interval (a, b) and is smooth inside. Also, it is easy to check that it satisfies

$$X' = \lambda - TX + X^2.$$

We already mentioned this fact in the previous section. From the equation, it is clear that X' goes to $+\infty$ at both ends of the interval.

We first prove the inequality $X' > 0$ on (a, b). To this end, consider the point x_0 where $v(x_0) = 0$. At this point, we also have $X = 0$ and therefore $X' = \lambda > 0$. We prove then our inequality separately on the 2 subintervals (a, x_0) and (x_0, b).

Consider first the subinterval (a, x_0). At both endpoints of this interval, the inequality holds true. Now, in a point where X' reaches 0, if such exists, the derivative X'' is

$$X'' = -T'X.$$

Since neither T' or X may change sign on (a, x_0), we obtain a contradiction either at the first passage of X' in 0, either at the last one.

Next, we turn to the second inequality

$$X' > \frac{n}{n-1} T',$$

which has only to be proved in the case where $T' > 0$ on our interval, by the previous result.

Once again, this inequality is true at the boundaries a and b. Let F denote

$$X' - \frac{n}{n-1} T'.$$

We first prove that $F \geqslant 0$ on (a, b). If this were not the case, then, considering the first passages of F below 0, there exists two points $x_1 < x_2$ in (a, b), with

$$F(x_1) = F(x_2) = 0; \qquad F'(x_1) \leqslant 0; \qquad F'(x_2) \geqslant 0; \qquad F < 0 \quad \text{on } (x_1, x_2).$$

Since F is the derivative of

$$X - \frac{n}{n-1} T,$$

then, we get

$$\left(X - \frac{n}{n-1} T\right)(x_2) < \left(X - \frac{n}{n-1} T\right)(x_1). \tag{16}$$

On the other hand, the equation satisfied by X shows that

$$X'' = X'[2X - T] - T'X,$$

such that in x_1 and x_2, where $\frac{n}{n-1} T' = X'$, one has

$$X'' = \frac{n}{n-1} T' \left[\frac{n+1}{n} X - T\right].$$

But, if we remember the definition of T, we have $T''=2TT'/(n-1)$, such that, in x_1 and x_2,

$$X''-\frac{n}{n-1}\,T''=F'=\frac{n}{n-1}\,T'(n+1)\left[X-\frac{n}{n-1}\,T\right].$$

Therefore, using the sign conditions on $F'(x_1)$ and $F'(x_2)$, we get

$$\left(X-\frac{n}{n-1}\,T\right)(x_1)\leqslant 0 \qquad \text{and} \qquad \left(X-\frac{n}{n-1}\,T\right)(x_2)\geqslant 0,$$

and this gives a contradiction to the inequality (16).

To check that the inequality is strict, we now know that whenever F reaches 0, then F' is 0, and at such a point, we get by the previous computation $X=(n/(n-1))\,T$. But the condition $X=(n/(n-1))\,T$ and the differential equation satisfied by X gives

$$X'=\lambda+\frac{n}{(n-1)^2}\,T^2,$$

while the identity $X'=(n/(n-1))\,T'$ leads to

$$X'=\frac{n}{n-1}\,R+\frac{n}{(n-1)^2}\,T^2.$$

This gives $\lambda=(n/((n-1))\,R$, which was excluded.

Notice that this is the first place where we really use the explicit form of the function $T(x)$, under the differential equality $T''=2TT'/(n-1)$, which makes no distinction between the positive and negative curved cases.

The next step is more technical, but is the key point of the next section: we only consider the case $n<\infty$.

PROPOSITION 3. *With the same notations as before, let $H(x)$ denote the function*

$$H(x)=\frac{n+2}{n}\,X\left(X-\frac{n}{n-1}\,T\right)+\lambda-\frac{n}{n-1}\,R.$$

Then $H(x)>0$.

Proof. Observe first that the inequalities are true at the boundaries a and b of the interval. Then, at the point x_0, $v=0=X$, and the inequality is also true since $\lambda > (n/(n-1))\, R$. We therefore have just to prove it separately on the intervals (a, x_0) and (x_0, b).

Using the same method as before, we compute H' at a point where $H = 0$. Using the differential equation satisfied by X and T, we obtain, at such a point

$$H' = -\frac{\lambda - \dfrac{n}{n-1}\, R}{X} \left[\frac{4\lambda}{n+2} + \frac{n-2}{n+2}\, \frac{n}{n-1}\, R \right].$$

This last expression keeps a constant sign on both the intervals (a, x_0) and (x_0, b). Therefore, the expression H cannot go below 0 on any of those intervals.

Remark. It may be true that for some values of λ, R, n, the previous expression is in fact 0, and therefore our conclusion slightly more delicate to derive. If such happens, then the conclusion holds true for any value λ' close to the one we considered, and we get the conclusion by a limiting argument.

Also, when $n = \infty$, it is clear from what precedes that the inequality remains true in the limit: in this case,

$$X(X-T) + \lambda - R > 0.$$

PROPOSITION 4. *Using the same notations as before, let*

$$Q_1(f) = -(f-T) \left[f - 2 \left(\frac{n}{n-1}\, T - X \right) \right]$$

and

$$Q_2(f) = -f \left(\frac{n-2}{2(n-1)}\, f - T + X \right).$$

Then, there exists a solution of the differential equation

$$f' = \min\{ Q_1(f), Q_2(f) \},$$

which remains finite on the interval (a, b). More precisely, the solution of this equation with initial condition $f(x_0) = 2T(x_0)$ does not explode on the full interval (a, b).

Proof. In this proposition, we are just concerned with the nonexplosion of the solution f of this equation inside the interval (a, b), but not with the nonexplosion problem at the boundaries a and b. For simplicity, we shall denote $T(x_0)=T_0$.

Since $\min\{Q_1(f), Q_2(f)\}$ goes to $-\infty$ when $f\to\pm\infty$, the solution starting from any value f_0 in x_0 cannot explode to $+\infty$ on (x_0, b) and neither explodes to $-\infty$ on (a, x_0).

Define the two functions y_1 and y_2 by

$$y_1=2\left(T-\frac{n-1}{n}X\right), \qquad y_2=2T.$$

Observe that $y_1<y_2 \Leftrightarrow x>x_0$. The first basic observation is that

$$Q_2(f)-Q_1(f)=\frac{n}{2(n-1)}(f-y_1)(f-y_2),$$

and therefore

$$Q_2(f)\leqslant Q_1(f)\Leftrightarrow \min\{y_1, y_2\}\leqslant f\leqslant \max\{y_1, y_2\}.$$

Observe also that the initial value we used for the solution is such that, in x_0, $f=y_1=y_2$, and $Q_1(f)=Q_2(f)$.

We first look at the asymptotic expansion of f at x_0: let $x=x_0+\varepsilon$. Then

$$f(x)=2T_0+2T_0^2/(n-1)\varepsilon+o(\varepsilon),$$
$$y_1(x)=2T_0+2T_0^2/(n-1)\varepsilon-2\frac{n-1}{n}\left(\lambda-\frac{nR}{n-1}\right)\varepsilon+o(\varepsilon),$$

and therefore, there exists some $\varepsilon>0$ such that $y_1<f$ on $(x_0, x_0+\varepsilon)$, and that $f<y_1$ on $(x_0-\varepsilon, x_0)$.

The next step is to prove that the curve $x\to f(x)$ cannot cross the curve $x\to y_1(x)$ at any other point than x_0. This shows that $f>y_1$ on (x_0, b) and that $f<y_1$ on (a, x_0), which completes the proof.

To see this, let us work first on the interval (x_0, b). Let us denote by $A(x)$ the quantity $f-y_1$. Suppose that our assertion were not true and consider the first point $x_1>x_0$ where $A(x_1)=0$. In a left neighbourhood of x_1, we know that $f'=Q_2(f)$, and therefore, by a simple computation,

$$A'(x_1)=2\frac{n-1}{n^2}\left[(n+2)X\left(X-\frac{n}{n-1}T\right)+n\left(\lambda-\frac{n}{n-1}R\right)\right]$$
$$=2\frac{n-1}{n}H(x_1).$$

But $A'(x_1) \leqslant 0$ since x_1 is the first 0 of A, and this is a contradiction with Proposition 3. The same argument holds on the interval (a, x_0) using the last passage of A at 0.

As a direct consequence of the previous proposition, we get the following:

COROLLARY 3. *For any compact subinterval* $[c, d]$ *of* (a, b), *there exists a smooth bounded function* f *on* $[c, d]$ *such that*

$$f' < \min\{Q_1(f), Q_2(f)\}.$$

Proof. For $\varepsilon > 0$, let us solve the differential equation

$$f'_\varepsilon = \min\{Q_1(f_\varepsilon), Q_2(f_\varepsilon)\} - \varepsilon,$$

with initial value $f_\varepsilon(x_0) = 2T(x_0)$, as before. Then, by standard perturbation theory of ordinary differential equations, the solution of this differential equation converges to the solution of the previous equation, uniformly on compact subsets of (a, b), when $\varepsilon \to 0$. This gives the result.

5. COMPARISON THEOREM FOR GRADIENTS OF EIGENFUNCTIONS

In this section, we consider an elliptic differential operator L on a compact manifold M, and we suppose that L satisfies the $CD(R, n)$ inequality, for some $R \in \mathbb{R}$ and $n \in (1, \infty)$. Our basic example L will be the Laplace–Beltrami operator on a compact Riemannian manifold, and, as we already mentioned, n will be in this case the geometric dimension, the Ricci curvature being bounded below by R. Everything extends easily to manifolds with smooth convex boundaries, provided that we restrict our attention to the Neumann boundary conditions.

Let u be a (bounded) real eigenfunction of L: this means that there is a constant λ such that

$$Lu = -\lambda u,$$

then the maximum principle yields that $\lambda \geqslant 0$.

The aim of this chapter is to describe the best possible upper bound of the form $\phi(u)$ for the (square of the) gradient $|\nabla u|^2 = \Gamma(u, u)$ (for simplicity, denote it by Γ) under the $CD(R, n)$ condition. This means that we look for a function ϕ from $[\min u, \max u]$ to $\mathbb{R}^+$ such that $\Gamma(u, u) \leqslant \phi(u)$.

As before, we shall consider our one dimensional model operator $L_{R,n}$ described before, and a solution v of the equation $L_{R,n}(v) = -\lambda v$ on some interval (a, b), with $v'(a) = v'(b) = 0$, $v' > 0$ on (a, b). Therefore, v is a solution of the one dimensional problem with Neuman boundary conditions associated with the same eigenvalue λ. The main result of this section is the following:

THEOREM 8. *If a solution v of the one dimensional problem with the same eigenvalue λ is such that* $[\min(u), \max(u)] \subset [\min(v), \max(v)]$, *then*

$$\Gamma(u, u) \leqslant (v' \circ v^{-1})^2 \, (u).$$

Observe that the inequality in Theorem 8 applies in particular to v itself, for which it is an equality. This is why this theorem is in fact a gradient comparison theorem, and therefore this result is optimal. Observe also that, thanks to the change of variable formula, this inequality may also be rewritten as

$$\Gamma(v^{-1} \circ u, v^{-1} \circ u) \leqslant 1,$$

which is the form we shall use it in the next part, and already shows the link between diameter (defined from functions h with $\Gamma(h, h) \leqslant 1$) and the eigenvector u.

Proof. First, since $\min(u) < 0$ and $\max(u) > 0$, we may change u into $u_1 = cu$ with $0 < c < 1$, in order to have $[\min(u_1), \max(u_1)] \subset (\min(v), \max(v))$, and then go to the limit when $c \to 1$. Therefore, we may assume that

$$[\min(u), \max(u)] \subset (\min(v), \max(v)).$$

Under these conditions, v^{-1} is a smooth function, with bounded derivatives of any order, on a neighbourhood of the interval $[\min(u), \max(u)]$.

Our strategy to prove an inequality of the form $\Gamma(u, u) \leqslant \phi(u)$ is to use an auxiliary smooth bounded function $\psi(x)$, strictly positive, and defined on a neighbourhood of the interval $[\min(u), \max(u)]$, and consider the function $F = \psi(u)[\Gamma(u, u,) - \phi(u)]$, which is a smooth bounded function defined on our manifold M. Then, we consider a point $p \in M$ where F attains its maximum value: such a point exists by compactness, and we shall prove that, for some adequate functions ψ, $F(p) \leqslant 0$, which gives the result. The function $\phi(u)$ is here strictly positive, and, at a maximum point, there is nothing to prove if $\Gamma = 0$. Then, we may assume $\Gamma(u, u)(p) > 0$.

The first remark is that there is a maximum point p such that $\nabla F(p)=0$. This is clear if the manifold has no boundary or p lies inside the interior of M. When the boundary is convex and p lies on the boundary, we may use the same argument as in [37]: the normal outer derivative of $\Gamma(u)$ is exactly $-2\pi(\nabla u, \nabla u)$ (π being the second fundamental form) since the normal derivative of u vanishes, and therefore it is nonpositive as π is positive. On the other hand, since p is a maximum point, the normal derivative of F at p is nonnegative. But a simple inspection shows that this derivative has the same sign as that of the normal outer derivative of $\nabla\Gamma$. Therefore, the normal derivative of Γ (and hence of $F(u)$) is 0, and the same being true for the tangential derivative, the conclusion follows.

To prove the result, we shall use the following lemma:

LEMMA 1. *Under the $CD(R, n)$ hypothesis, and for any smooth functions $\phi(x)$ and $\psi(x)>0$, and at a maximum point p of F such that $\Gamma(p)\neq 0$, we have*

$$\begin{aligned} 0\geqslant LF \geqslant & \frac{1}{\psi}\left[\frac{\psi''}{\psi}-\left(2-\frac{n}{2(n-1)}\right)\frac{\psi'^2}{\psi^2}\right]F^2 \\ & +\left[\frac{\psi''}{\psi}\,\phi-2\phi\,\frac{\psi'^2}{\psi^2}-\frac{n}{n-1}\,\phi'\,\frac{\psi'}{\psi}-\frac{n+1}{n-1}\,\lambda u\,\frac{\psi'}{\psi}\right]F \\ & +(-2\lambda-\phi''+2R)\,F+\phi\psi(-2\lambda-\phi''+2R) \\ & +\lambda u\psi\phi'+\frac{2}{n}\,\lambda^2u^2\psi+\frac{2n}{n-1}\,\psi\left(\frac{\lambda u}{n}+\frac{\phi'}{2}\right)^2, \end{aligned} \tag{17}$$

We remind the reader that here Γ stands for $\Gamma(u, u)$.

Proof (of Lemma 1). First, the inequality $0\geqslant LF(p)$ just comes from the maximum principle. Now, the standard chain rule shows that, for any smooth function $\Psi(x, g)$,

$$\begin{aligned} L(\Psi(u,\Gamma)) = {} & \Psi'_1(u,\Gamma)\,Lu+\Psi'_2(u,\Gamma)\,L\Gamma \\ & +\Psi''_{11}(u,\Gamma)\,\Gamma+2\Psi''_{12}(u,\Gamma)\,\Gamma(u,\Gamma)+\Psi''_{22}(u,\Gamma)\,\Gamma(\Gamma,\Gamma). \end{aligned} \tag{18}$$

For the function $\Psi(u, g)=\psi(u)[g-\phi(u)]$, we have $\Psi''_{22}=0$, and the last term vanishes. Also, in this formula, we may replace Lu by $-\lambda u$ and $L\Gamma$ by

$$L\Gamma(u,u)=2\Gamma_2(u,u)+2\Gamma(u, Lu)=2\Gamma_2(u,u)-2\lambda\Gamma.$$

Now, we also know that at the maximum point p, $\nabla F=0$, which may be rewritten as

$$\psi'(u)[\Gamma-\phi(u)]\ \nabla u+\psi(u)[\nabla\Gamma-\phi'(u)\ \nabla u]=0.$$

This gives

$$\psi\Gamma(u,\Gamma)=-\frac{\psi'}{\psi}\ F\Gamma+\psi\phi'\Gamma. \tag{19}$$

Now, if we insert these identities into Eq. (18) for our function $\Psi(u, g)$, we get, at this point p,

$$\begin{aligned} LF=&\left(\frac{\psi''}{\psi}-2\ \frac{\psi'^2}{\psi^2}\right) F\Gamma-\lambda u\ \frac{\psi'}{\psi}\ F \\ &+\psi(-2\lambda\Gamma-\phi''\Gamma+\lambda u\phi')+2\psi\Gamma_2, \end{aligned} \tag{20}$$

where Γ_2 stands here for $\Gamma_2(u, u)(p)$.

On the other hand, if we apply the extended curvature-dimension inequality (7) to the function u, we get

$$\begin{aligned} \Gamma_2\geqslant&\frac{1}{n}\ \lambda^2u^2+R\Gamma \\ &+\frac{n}{n-1}\left(\frac{\lambda u}{n}+\frac{\Gamma(u,\Gamma)}{2\Gamma}\right)^2. \end{aligned}$$

We may again use the identity given by (19) and the fact that $\Gamma\neq 0$ at the point p to obtain

$$\begin{aligned} 2\psi\Gamma_2\geqslant&\frac{n}{2(n-1)}\ \frac{\psi'^2}{\psi^3}\ F^2-\frac{2n}{n-1}\ \frac{\psi'}{\psi}\left(\frac{\lambda u}{n}+\frac{\phi'}{2}\right) F \\ &+\psi\left[2R\Gamma+\frac{2}{n}\ \lambda^2u^2+\frac{2n}{n-1}\left(\frac{\lambda u}{n}+\frac{\phi'}{2}\right)^2\right]. \end{aligned} \tag{21}$$

Now, we just have to use the inequality (21) to replace the term Γ_2 in (20), and replace everywhere Γ by $\phi+F/\psi$ to obtain our result.

Notice that the left-hand side of the inequality given by Lemma 1 is a second-order polynomial expression in the variable F.

Let us apply this inequality with the function $\phi(x)=(v'\circ v^{-1})^2$. If we do this on our model space with the function v instead of u, we have of course $F=0$. But, in this case, the extended curvature dimension inequality is an equality, as we already noticed in Section 2. Hence, the inequality given by

Lemma 1 is an equality, and therefore the 0 order term in the polynomial has to be 0: this explains why, when $\phi(x)=(v'\circ v^{-1})^2$, then we have

$$\phi(-2\lambda-\phi''+2R)+\lambda x\phi'+\frac{2}{n}\lambda^2x^2+\frac{2n}{n-1}\left(\frac{\lambda x}{n}+\frac{\phi'}{2}\right)^2=0. \qquad (22)$$

Of course, one may obtain this inequality directly from the equations

$$v''=Tv'-\lambda v, \qquad \text{and} \qquad T'=R+T^2/(n-1).$$

In fact, it appears that the different solutions of the second-order differential equation given by (22) are just the functions $(v'\circ v^{-1})^2$ on different intervals, with different boundary values.

Now, in order to prove our main result, we use the function $\phi(x)=(v'\circ v^{-1})^2\,(x)$ as before, and write $\psi=\exp(g(x))$. At a maximum p of F such that $F(p)>0$, we have

$$0\geqslant F\left[\frac{1}{\psi}\left(g''-\frac{n-2}{2(n-1)}g'^2\right)F\right.$$
$$\left.+(g''-g'^2)\,\phi-\frac{n}{n-1}\phi'g'-\frac{n+1}{n-1}\lambda ug'-2\lambda+2R-\phi''\right]. \qquad (23)$$

Define two functions T_1 and T_2 by

$$T_1=g''-\frac{n-2}{2(n-1)}g'^2, \qquad (24)$$

$$T_2=(g''-g'^2)\,\phi-\frac{n}{n-1}\phi'g'-\frac{n+1}{n-1}\lambda xg'-2\lambda+2R-\phi''. \qquad (25)$$

At the point p, we have

$$F\left(\frac{T_1}{\psi}F+T_2\right)\leqslant 0.$$

If we find a function g on $[\min(u),\max(u)]$ such that $T_1>0$ and $T_2>0$ on our interval, then we would have

$$0<F\left(\frac{T_1}{\psi}F+T_2\right)\leqslant 0$$

which is impossible. Therefore we may conclude that $F\leqslant 0$, hence $\Gamma(u,u)\leqslant\phi(u)$.

Let $h = g'$. As in the previous section, we set $X = \lambda \frac{v}{v'}$. The differential equation satisfied by ϕ gives

$$-2\lambda + 2R - \phi''(v) = -2T \left(\frac{n}{n-1} T - X \right).$$

Define the function f by $f(x) = -h(v)\, v'$. Using again the differential equation satisfied by v, we get

$$f' = -h'(v)\, v'^2 + Tf - Xf.$$

Now, with these notations, we have

$$\begin{aligned} T_2 &= -f' - f^2 - f\left(2X - \frac{3n-1}{n-1} T\right) - 2T\left(\frac{n}{n-1} T - X\right) \\ &= Q_1(f) - f'. \end{aligned}$$

Also,

$$\begin{aligned} v'^2 T_1 &= -f' - \frac{n-2}{2(n-1)} f^2 - f(T - X) \\ &= Q_2(f) - f'. \end{aligned}$$

We may now use Corollary 3 of the previous section: there exists a bounded function f on $[\min(u), \max(u)] \subset (\min(v), \max(v))$ which satisfies $f' < \min(Q_1(f), Q_2(f))$. Then, any primitive of the function $-\frac{f}{v'} \circ v^{-1}$ on the interval $[\min(u), \max(u)]$ is a function g for which both T_1 and T_2 are positive. The proof is completed.

As a consequence, we may apply the result to the different functions v solutions to our eigenvalue problems in one dimension. We get the corollary

COROLLARY 4. *Let T_i $(i = 1, 2)$ be two functions satisfying the same differential equation $T' = R + \frac{T^2}{n-1}$, for some $R \in \mathbb{R}$ and $n \in (1, \infty]$ on two different intervals $[a_i, b_i]$. Then, let v_i be the solutions to the differential equations*:

$$v_i'' - T_i v_i' = -\lambda v_i, \qquad \text{on } (a_i, b_i)$$

with Neumann boundary conditions $v_i'(a_i) = v_i'(b_i) = 0$ *and* $v_i' \neq 0$ *on* (a_i, b_i). *Let* m_i *and* M_i *be the minimum and the maximum of* v_i *on* (a_i, b_i), $i = 1, 2$, *respectively. If* $[m_1, M_1] \subset [m_2, M_2]$, *then*

$$|v_1' \circ v_1^{-1}| \leqslant |v_2' \circ v_2^{-1}|, \qquad \text{on } [m_1, M_1].$$

In particular, if $x_i \in (a_i, b_i)$ *such that* $v_1(x_1) = v_2(x_2)$, *then* $|v_1'(x_1)| \leqslant |v_2'(x_2)|$.

As a consequence of this corollary, we may now prove a result which we already mentioned in section 3: for the Neumann eigenvector of the one-dimensional model, the ratio $-\max(v)/\min(v)$ is in fact an increasing function of the left end point of the interval when T is increasing, and a decreasing function when T is decreasing. More precisely, we have:

COROLLARY 5. *Let* I *denotes an interval on which the function* T *satisfies* $T' = R + (T^2/(n-1))$, $n > 1$. *Assume that for two points* $a_1 < a_2$ *in* I, *the functions* v_i, $i = 1, 2$ *are solutions of the differential equation*

$$v'' = Tv' - \lambda v,$$

with initial value $v_i(a_i) = -1$, $v_i'(a_i) = 0$. *Define* b_i *by*

$$b_i = \inf\{x > a_i \mid v_i'(x) = 0\},$$

and assume that both b_1 *and* b_2 *belong to* I. *Then, if* T *is increasing in* I, $v_2(b_2) \geqslant v_1(b_1)$, *while if* T *is decreasing on* I, $v_1(b_1) \geqslant v_2(b_2)$.

Proof. We deal only with the case where T is increasing, the other cases are completely similar. For any starting point a in $[a_1, a_2]$, let us denote v_a the eigenvector

$$v_a'' = Tv_a' - \lambda v_a, \qquad v_a(a) = -1; \qquad v_a'(a) = 0.$$

By the nonoverlapping property of the Neumann eigenvectors, we know that

$$b(a) = \inf\{x > a \mid v_a'(x) = 0\}$$

lies inside I and more precisely inside $[b(a_1), b(a_2)]$. We want to prove that the function $m(a) = v_a(b(a))$ is increasing with T.

For that, observe first that all those quantities are smooth functions of a inside the interval I. More precisely, since all the functions v_a for different a's are solutions of the same second-order linear differential equation, then we may write $v_{a+\varepsilon}$ as a linear combination of the function v_a and of the function w_a solution of

$$w_a'' = Tw_a' - \lambda w_a, \qquad w_a(a) = 0, \qquad w_a'(a) = 1.$$

If we do that, and if we go to the limit when ε goes to 0, we get

$$\frac{\partial v_a}{\partial a} = -\lambda w_a, \text{ and} \tag{26}$$

$$\frac{\partial w_a}{\partial a} = v_a - T(a)\, w_a. \tag{27}$$

From this we get that

$$m'(a) = -\lambda w_a(b(a)).$$

Now, assume that, for some point a_0 inside the interval, we had $m'(a_0) < 0$. Then, in some right neighbourhood of a_0, we would have $m(a) < m(a_0)$, and then we could apply the gradient comparison theorem (Corollary 4) and get that, for any $a > a_0$ close enough to a, and for any $x \in [-1, b(a)]$, we have

$$v_a' \circ v_a^{-1}(x) \leqslant v_{a_0}' \circ v_{a_0}^{-1}(x).$$

Letting $a \to a_0$, $b(a) \to b(a_0)$ we get in the limit

$$\forall x \in (-1, b(a_0)), \qquad \frac{\partial}{\partial a}\, v_{a_0}' \circ v_{a_0}^{-1}(x) \leqslant 0.$$

However,

$$\frac{\partial}{\partial \alpha}\, v_a^{-1}(x) = -\frac{\dfrac{\partial}{\partial a}\, v_a}{\dfrac{\partial}{\partial x}\, v_a}\, (v_a^{-1}),$$

and therefore

$$\frac{\partial}{\partial a}\, \frac{\partial}{\partial x}\, v_a\, \frac{\partial}{\partial x}\, v_a \leqslant \frac{\partial^2}{\partial x^2}\, v_a\, \frac{\partial}{\partial a}\, v_a.$$

Using (26) and (27), we get

$$w_{a_0} v_{a_0}'' - w_{a_0}' v_{a_0}' \leqslant 0$$

on the whole interval $[a_0, b(a_0)]$. But an asymptotic expansion of $H_a(x) = w_a v_a'' - w_a' v_a'$ in $x = a + \varepsilon$ gives

$$H_a(x+\varepsilon) = \lambda T'(a)\, \varepsilon^3/3 + o(\varepsilon^3) > 0,$$

and this gives a contradiction.

Remark. As a consequence of the preceding result, we get that $w_a(b(a)) < 0$, which shows that λ is the eigenvalue of the Dirichlet problem corresponding to some smaller interval (namely $[a, c(a)]$, where $c(a)$ is the first 0 of w_a). This shows in particular that the Dirichlet eigenvalue for $[a, b]$ is smaller than the Neumann spectral gap, when T is increasing.

Another interesting consequence is the eigenvalue comparison theorem when the eigenvector has a minimum value opposite to it's maximum value: we shall use this first step later on

COROLLARY 6. *Suppose that the operator L satisfies $CD(R, n)$ and that the diameter is bounded above by δ. Then, if u is a solution of $L(u) = -\lambda u$, (with Neumann boundary conditions if there is a convex boundary), and if* $\min(u) = -\max(u)$, *then $\lambda \geqslant \lambda_0$, where λ_0 is the Neumann eigenvalue for the operator $L_{R,n}$ on $[-\delta/2, \delta/2]$.*

Proof. We may as well suppose that $\min(u) = -1$, $\max(u) = 1$. For the operator $L_{R,n}$, we may find a symmetric interval $[-\delta_0/2, \delta_0/2]$ on which $L_{R,n}$ has eigenvalue λ. Let v be the corresponding eigenvector.

The operator $L_{R,n}$ is invariant under $x \to -x$, and therefore if $\min(v) = v(-\delta_0/2) = -1$, then $\max(v) = v(\delta_0/2) = 1$, and we may use Theorem 8 to get $|\nabla g| \leqslant 1$, where g is the function $g = v^{-1} \circ u$. Now, let p_1 and p_2 two points such that $u(p_1) = -1$, $u(p_2) = 1$. By definition of the distance function,

$$\delta \geqslant d(p_1, p_2) \geqslant g(p_2) - g(p_1) = \delta_0.$$

Therefore, since the eigenvalue of $[-d, d]$, for the operator $L_{R,n}$ is a decreasing function of d, and that $d = \delta_0$ corresponds to λ, the eigenvalue of $[-\delta/2, \delta/2]$ is smaller than λ.

As a consequence of the estimates given in the next sections, we shall get the extension of this result to the general case.

6. COMPARISON THEOREM FOR THE MAXIMA OF EIGENFUNCTIONS

The main purpose of this section is to prove a comparison theorem on the maximum values of eigenvectors, under lower bounds on the Ricci curvature and upper bounds on the dimension. As in the previous section, we work with a second-order differential operator L acting on a compact manifold M (with or without convex boundary), and satisfying the $CD(R, n)$ hypothesis. In all this chapter, the dimension parameter n is finite. The results of this section shall not be used in the case of a $CD(R, \infty)$ dimension.

This result will be obtained using a standard property of the volume of small balls with respect to the invariant measure.

Up to now, we made little use of the invariant measure (the Riemannian measure in the case of Laplacians), which is nevertheless the essential tool used for the Bochner–Lichnerowicz bound on the spectral gap. Here, we shall use it, and we denote it by $\mu(dx)$. Since our manifold is compact, we may assume that the measure is finite, and also, since it is defined up to a multiplicative factor, we may as well assume that it is a probability (this has no real importance, and is just here to fix the ideas). Although we are mainly interested in the symmetric case (the operator L is symmetric in $L^2(\mu)$), we just need here that the measure μ is reversible, that is,

$$\forall f \text{ smooth and bounded}, \int_M L(f)\, \mu(dx) = 0. \tag{28}$$

In short, it means that, when $L = \Delta + B$, where Δ is the Laplacian of some Riemannian structure and B a vector field, we do not require B to be ∇h.

If our operator is elliptic with smooth coefficients, it is always true that μ has a smooth strictly positive density with respect to the Riemannian measure, and therefore, if $B(x, r)$ denotes the ball centred at x with radius r, and if p denotes the usual dimension of the manifold, then

$$\liminf_{r \to 0} \mu(B(x, r))\, r^{-p} \geqslant C(x), \tag{29}$$

for some $C(x)$ which is the density of the invariant measure with respect to the Riemannian measure.

Since the dimension n is always larger than p (and is p only in the case of Laplacians), we also have

$$\forall x \in M, \liminf_{r \to 0} \mu(B(x, r))\, r^{-n} > 0. \tag{30}$$

Notice that this liminf is in general infinite.

To simplify the idea, let us first reduce the number of cases under consideration. Since we work only with finite dimensions, our model operators here are $L_{R,n}$ with $R \in \mathbb{R}$. Also, in all the results of this chapter, the case $R=0$ may be obtained by letting $R \to 0$, $R<0$, since an operator satisfying $CD(0,n)$ satisfies also $CD(R,n)$ for every $R<0$. So we just consider the cases $R \neq 0$, and by the scaling property that we mentioned in Section 2, we may assume that $R = \pm(n-1)$. We shall work mainly with the case $R=(n-1)$, since the proofs are completely similar in the other case.

The main result of this section is the comparison of the ratio between maximum and minimum value of an eigenvector associated to L with corresponding ratio of the one-dimensional model associated with the same eigenvalue and on the bounded interval.

THEOREM 9. *Assume that the operator L satisfies $CD(n-1,n)$, and that u is an eigenvector solution of $L(u) = -\lambda u$, with $\lambda > n$. Assume that* $\min(u) = -1$. *Consider the solution v of the differential equation*

$$v'' = (n-1)\tan(x)\, v' - \lambda v;\ v(-\pi/2) = -1;\ v'(-\pi/2) = 0.$$

Let $b = \inf\{x > -\pi/2 \mid v'(x) = 0\}$ *and* $m = v(b)$. *Then* $\max(u) \geqslant m$.

We also get the equivalent results for the negative and null curvature cases.

THEOREM 10. *Assume that the operator L satisfies $CD(-(n-1),n)$ and that u is an eigenvector solution of $L(u) = -\lambda u$, with $\lambda > 0$. Assume that* $\min(u) = -1$. *Consider the solution v of the differential equation on* $(0,\infty)$

$$v'' = -(n-1)\coth(x)\, v' - \lambda v;\ v(0) = -1;\ v'(0) = 0.$$

Let $b = \inf\{x > 0 \mid v'(x) = 0\}$ *and* $m = v(b)$. *Then* $\max(u) \geqslant m$.

Remark. In view of the result of Corollary 5 of the previous section, the minimum value over all possible intervals of the maximum value of the one-dimensional eigenvector on $[a, b(a)]$ is attained when the drift T is minimum, since it is an increasing function of $T(a)$. Then, this minimum is obtained at the "left" boundary, when $T(a) = -\infty$.

Also, it may happen, as we already mentioned, that b is infinite and in this case m is infinite: this case is irrelevant.

Finally, we get also the "flat" comparison theorem:

THEOREM 11. *Assume that the operator L satisfies $CD(0,n)$ and that u is an eigenvector solution of $L(u) = -\lambda u$, with $\lambda > 0$. Assume that* $\min(u) = -1$. *Consider the solution v of the differential equation on* $(0, \infty)$

$$v'' = -\frac{(n-1)}{x} v' - \lambda v; \qquad v(0) = -1; \quad v'(0) = 0.$$

Let $b = \inf\{x > 0 \mid v'(x) = 0\}$ and $m = v(b)$. Then $\max(u) \geqslant m$.

As we already mentioned, we shall just prove the first case, the proofs in the other cases are completely similar. Therefore, in order to simplify the notations, we shall use the notation L_0 for the one-dimensional model operator

$$L_0(v) = v'' - (n-1)\tan(x)\, v'.$$

The proof relies on another comparison theorem. For this, consider any Neumann eigenvector v solution of $L_0(v) = -\lambda v$, on some subinterval $[a, b(a)] = [a, b]$ of $[-\pi/2, \pi/2]$, with $v(a) = -1$. Let x_0 be the unique 0 of v in $[a, b]$.

Assume that $v(b) = m \geqslant \max(u)$. Then, the function $g(x) = v^{-1} \circ u$ takes value in $[a, b]$, and we shall consider the image measure m of μ under g:

$$m(ds) = \mu(v^{-1}(u) \in ds).$$

By definition, this measure is such that, for any Borel bounded function f, we have

$$\int f(s)\, m(ds) = \int_M f(g(x))\, \mu(dx).$$

Formally, and by the co-area formula, this measure m has a density $m_0(s)$ with respect to ds which is

$$m_0(s) = \int_{\{v^{-1}(u) = s\}} |\nabla v^{-1}(u)|^{-1},$$

but it is not clear that such density exists under the general hypothesis of this section. We shall not need it. Then, we have

PROPOSITION 5. *Under the previous hypotheses, let $E(s)$ be*

$$E(s) = -\lambda \int_a^s v(r)\, m(dr) \exp\left(\lambda \int_{x_0}^s \frac{v}{v'}\, dt\right).$$

Then, $E(s)$ increases on $(a, x_0]$ and decreases on $[x_0, b)$.

As usual, we use the convention that $\int_x^s = -\int_s^x$ if $s < x$.

Remark. Notice that this is a comparison theorem. When $u = v$, then the measure $m(ds)$ is the invariant measure $\rho(s)\,ds$ of one-dimensional model (here $\rho(s) = C_n \cos(x)^{n-1}$). In fact, in this case, we have

$$v'' + \frac{\rho'}{\rho}\,v' = -\lambda v,$$

and therefore

$$-\lambda\,\frac{v}{v'} = \frac{v''}{v'} + \frac{\rho'}{\rho}, \qquad \text{and} \qquad (\rho v')' = -\lambda \rho v.$$

We get

$$-\lambda \int_a^s v(r)\,m(dr) = \rho(s)\,v'(s), \qquad \text{and} \qquad \exp\left(\lambda \int_{x_0}^s \frac{v}{v'}\,dt\right) = \frac{\rho(x_0)\,v'(x_0)}{\rho(s)\,v'(s)}.$$

Therefore, the expression $E(s)$ is constant in this case.

In the end, and if we remark that

$$\int_a^s v(r)\,m(dr) = \int_M u 1_{\{u \leqslant v(s)\}}\,\mu(dx),$$

we get the following measure comparison theorem, which is just a rewriting of Proposition 5:

THEOREM 12. *Suppose that L satisfies $CD(R, n)$, has invariant measure μ and that u satisfies $L(u) = -\lambda u$. Let $L_{R,n}$ be the one-dimensional model, on any interval, with invariant measure $\mu_{R,n}$, and suppose that v satisfies $L_{R,n}(v) = -\lambda v$, with the same λ. If $[\min(u), \max(u)] \subset [min(v), \max(v)]$, then the ratio*

$$R(c) = \frac{\int u 1_{\{u \leqslant c\}}\,d\mu}{\int v 1_{\{v \leqslant c\}}\,d\mu_{R,n}}$$

is increasing on $[\min(u), 0]$ and decreasing on $[0, \max(u)]$.

Proof (of Proposition 5). Choose any smooth positive function $H(s)$ compactly supported inside (a, b), and define the function G such that $(G \circ v)' = H(s)$, $G(-1) = 0$, where G is defined on $[-1, m = v(b)]$. Choose again a function $K(x)$ such that $K(x)+xK'(x)=G(x)$. We have, using the chain rule,

$$L(uK(u))=L(u)[K(u)+uK'(u)]+\Gamma(u,u)[uK''(u)+2K'(u)] \tag{31}$$

$$=G(u)\,L(u)+\Gamma(u,u)\,G(u). \tag{32}$$

By invariance,

$$\int_M L(uK(u))\,\mu(dx)=0,$$

and therefore

$$\int_M L(u)\,G(u))\,\mu(dx)=-\int_M G'(u)\,\Gamma(u,u)\,\mu(dx).$$

(This computation just shows us that, as long as we work only on functions of u, there is no difference between invariance and symmetry for our measure μ.)

Then, we may apply our gradient comparison theorem, which tells us that $\Gamma(u,u)\leqslant(v'\circ v^{-1})^2\,(u)$. Since $L(u)=-\lambda u$, the previous identity gives

$$\lambda\int_M uG(u)\,\mu(dx)\leqslant\int_M G'(u)(v'\circ v^{-1})^2\,(u)\,\mu(dx).$$

If we remind that $g=v^{-1}(u)$, we may write $u=v(g)$, and we get

$$\lambda\int_M v(g)\,G(v(g))\,\mu(dx)\leqslant\int_M G'(v(g))\,v'(g)^2\,\mu(dx).$$

Now, we use the fact that $m(ds)$ is the image measure of g to get

$$\lambda\int_a^b v(s)\,G(v(s))\,m(ds)\leqslant\int_a^b G'\circ v(s)\,v'^2(s)\,m(ds).$$

Since $(G\circ v)'=H$, we have

$$G'\circ v(s)\,v'^2(s)=H(s)\,v'(s),\qquad \text{while}\quad G\circ v(s)=\int_a^s H(r)\,dr.$$

This give, changing the integration order,

$$\int_a^b\left[\int_s^b \lambda v(r)\,m(dr)\right]H(s)\,ds\leqslant\int_a^b H(s)\,v'(s)\,m(ds). \tag{33}$$

This last equation holds true for any positive function H defined on the support of $m(ds)$. Also, observe that $\int_M u\, d\mu(x) = 0$ and therefore

$$\int_a^b v(r)\, m(dr) = 0 \Rightarrow \int_s^b v(r)\, m(dr) = -\int_a^s v(r)\, m(dr).$$

Let A be the function

$$A(s) = -\lambda \int_a^s v(r)\, m(dr)$$

it is continuous with bounded variation, and Eq. (33) tells us that the measure

$$v'(s)\, m(ds) - A(s)\, ds$$

is positive.

In other words, the measure $\lambda A ds + \frac{v'}{v}\, dA$ is negative, which tells us that the measure

$$\lambda \frac{v}{v'} A\, ds + dA$$

is positive on $[a, x_0]$ and negative on $[x_0, b]$. (v' is positive on $[a, b]$ and v is positive on $[x_0, b]$ and negative on $[a, x_0]$.) If

$$E(s) = A(s) \exp\left[\lambda \int_{x_0}^s \frac{v}{v'}(r)\, dr\right],$$

then it is also a continuous function with bounded variation on any compact subinterval of (a, b), and we have $dE(s) \leqslant 0$ on $[x_0, b)$ and $dE(s) \geqslant 0$ on $(a, x_0]$. This proves the proposition.

Remark. It is clear from the proof that the previous proposition is valid under any $CD(R, n)$ hypothesis, even with infinite dimension, provided we use the right operator $L_{R,n}$ and the right measure $\mu_{R,n}$. It relies only on the main gradient comparison theorem of the previous section.

For the eigenvector u associated with λ, under the hypothesis $CD(n-1, n)$, we get from the previous proposition the following:

COROLLARY 7. *Suppose* $\min(u) = -1$ *and suppose that* $\max(u) < m$, *where* m *is the maximum value of the boundary eigenvector for the model. Let* p *be a point where* u *is minimum. Then there exists a constant* $c > 0$ *such* *that, if r is small enough, the volume $\mu(B(p,r))$ of the ball with centre p and radius r is such that*

$$\mu(B(p,r)) \leqslant cr^n.$$

Proof. The proof is straightforward from Theorem 12. As mentioned above, we work to fix the ideas with the operator L_0, which is the model for $CD(n-1,n)$. Indeed, using the result of corollary 45 with $d=0$, and $c \leqslant -1/2$, we observe that

$$\mu(\{u \leqslant c\}) \leqslant 2 \int_{\{u \leqslant c\}} |u| \, d\mu \leqslant 2C \int_{\{v \leqslant c\}} |v| \, d\mu_0 \leqslant 2C\mu_0(\{v \leqslant c\}),$$

where C is the ratio

$$\frac{\int_{\{u \leqslant 0\}} |u| \, d\mu}{\int_{\{v \leqslant 0\}} |v| \, d\mu_0}.$$

Now, when $c = -1 + \varepsilon$, with $\varepsilon \to 0$, the set $\{u \leqslant c\}$ contains a ball with centre p and radius bounded by $C_1 \sqrt{\varepsilon}$: this comes from the fact that p is a singular point, and the constant p may be controlled by the maximum of the Hessian of u in a neighbourhood of p; if $d(p,x) \leqslant C_1 \sqrt{\varepsilon}$, then $u(x) \leqslant u(p) + \epsilon$. (This argument is valid even if the minimum is on the boundary, when such exists, since u satisfies Neumann boundary conditions.)

For the same reason, since we have a lower bound of $|v''|$ in a neighbourhood of $-\pi/2$, $(v''(-\pi/2) = -\lambda/n)$, we know that the set $\{v \leqslant c\}$ is included in an interval of length $C_2 \sqrt{\varepsilon}$, and $\mu_0([-\pi/2, -\pi/2+\alpha])$ behaves like α^n when $\alpha \to 0$. We get the result by putting all these arguments together.

We may now come to the proof of the maxima comparison theorem.

Proof (of Theorem 9). Assume that L satisfies $CD(n-1,n)$, and let u be a solution of the equation $L(u) = -\lambda u$, with $\min(u) = -1$.

For the model operator

$$L_{R,n}(v) = v'' - \sqrt{(n-1)\, R} \tan\left(\sqrt{\frac{R}{n\text{-}1}}\, x\right) v',$$

we consider the boundary solution v

$$L_{R,n}(v) = -\lambda v, \; v\left(-\pi/2 \sqrt{\frac{n\text{-}1}{R}}\right) = -1; \qquad v'\left(-\pi/2 \sqrt{\frac{n\text{-}1}{R}}\right) = 0,$$

and its maximum value $m(R, n)$, which is $v(b)$, where

$$b = \inf\left\{x > -\pi/2\sqrt{\frac{n\text{-}1}{R}}\ \middle|\ v'(x) = 0\right\}.$$

Then, clearly, $m(R, n)$ is a continuous function of (R, n), in a neighbourhood of $(n-1, n)$.

Now, assume that $\max(u) < m(n-1, n)$. Then, certainly, we also have $\max(u) < m(n-1, n')$, for any $n' > n$ close enough to n. But, if L satisfies $CD(R, n)$, it also satisfies $CD(R, n')$, for the same R and any $n' > n$. Therefore, for $n' > n$ and close enough to n, we may apply corollary 48, and we get, for some constant C

$$\mu(B(p, r)) \leqslant Cr^{n'},$$

for r going to 0. This contradicts to (30). The proof is completed.

Remark. The common feature of the three cases of positive, negative, and 0 curvature for our models with finite dimension, is that, at the boundary (and only there), the measure of the ball of radius r behaves as Cr^n when $r \to 0$, as we already mentioned. At any other points, the volume of balls behave as Cr, as is usual for a one-dimensional model. This peculiar behaviour comes from the singularity of the drift term at the boundary, which has the same form in those three models. It is at the boundary, and only there, that the notion of dimension in our model operators coincides with the notion which comes from the volume of balls. This explains the particular role played by the boundary solutions in this analysis.

7. THE WORST CASE IN ONE-DIMENSIONAL MODELS

In this chapter, we prove the last ingredient of our main result. For this, we come back to our model operators $L_{R,n}$. Except in the case of $L_{0,\infty}$ (which may be dealt with separately since everything may then be computed explicitly), all these operators are defined on symmetric "intervals," even though in the case of $L_{-(n-1),n}$ this interval is bigger than $\mathbb{R}$ itself. We already know that the diameter δ is bounded above by $\delta_0 = \pi\sqrt{R/(n-1)}$ when $R > 0$. So we shall consider model spaces of length $\delta < \delta_0$, with $\delta_0 = \pi\sqrt{R/(n-1)}$ if $R > 0$, $n < \infty$, and $\delta_0 = \infty$ in all the other cases.

On any subinterval of finite length $\delta < \delta_0$, say $[a, a+\delta]$, the operator $L_{R,n}$ may be considered as a model operator for $CD(R, n)$, with diameter δ. Any of those models have a Neumann eigenvalue $\lambda(R, n, \delta, a)$ and our problem is to determine which one has the lowest eigenvalue, when a varies among all the possible values. The answer is the following:

THEOREM 13.

$$\forall n \in (1, \infty], \qquad R \in \mathbb{R}, \qquad \delta < \delta_0, \qquad \lambda(R, n, \delta, a) \geqslant \lambda(R, n, \delta, -\delta/2).$$

In other words, the central interval has the lowest Neumann eigenvalue.

Although this looks like a simple one-dimensional problem, the proof given below relies again on the gradient comparison theorem of Section 5 and makes full use of the Γ_2 techniques in one dimension.

Proof. To prove Theorem 13, we shall use the fact that when R, n, a are fixed, λ is a decreasing function of δ. First, by symmetry, it is enough to work on intervals with $a < -\delta/2$. Then, let $\lambda = \lambda(R, n, \delta, -\delta/2)$ be the eigenvalue corresponding to the central interval, and, for any $a < -\delta/2$, let us solve the Neumann eigenvector problem starting from a:

$$L_{R,n}(v_a) = -\lambda v_a; \qquad v_a'(a) = 0; \qquad v_a(a) = -1.$$

Consider the end point of the interval:

$$b(a) = \inf \{x > a \mid v_a'(x) = 0\}.$$

By the nonoverlapping property of intervals sharing the same eigenvalue, we know that $b(a) < \delta/2$. If $\delta_0(a)$ denotes the length $b(a) - a$, then we have

$$\lambda(R, n, \delta, -\delta/2) = \lambda = \lambda(R, n, \delta_0(a), a).$$

Therefore,

$$\lambda(R, n, \delta, a) \geqslant \lambda(R, n, \delta, -\delta/2) \Leftrightarrow \delta_0(a) \geqslant \delta.$$

For this, we shall in fact use the gradient comparison Theorem 8, and more precisely Corollary 6.

In what follows, we fix a, and write b for $b(a)$, c for $\delta_0(a)/2$. We then consider a solution v of the eigenvalue problem for $L_{R,n}$ on $[a, b]$, and set $m = (a+b)/2$. We move everything by translation to a central interval, such that $v(x) = v(m+y) = v_1(y)$, where $y \in [-c/2, c/2]$, and v_1 satisfies

$$v_1'' = T_1 v_1' - \lambda v_1,$$

with $T_1(y) = T(m+y)$, $v_1(-c) = -1$, $v_1'(-c) = v_1'(c) = 0$.

Consider also the symmetric interval $[-b, -a]$: it has of course the same eigenvalue for $L_{R,n}$ and $w(x) = -v(-x)$ is the symmetric solution of the eigenvalue problem, with final value $w(-a) = -1$, $w'(-a) = 0$. We also move it on the symmetric interval $[-c, c]$, and get a solution of the eigenvalue problem

$$v_2'' = T_2 v_2' - \lambda v_2,$$

where $T_2(y) = T(-m+y) = -T(m-y)$. Moreover, we have

$$v_2(-c) = -v_1(c), \qquad v_2(c) = -v_1(-c).$$

Observe that both T_1 and T_2 are solutions of the differential equation $T' = R + T^2/(n-1)$, since it is invariant under translation.

Now, consider the function $V = v_1 + v_2$. It is the solution of the eigenvalue problem

$$V'' = T_0 V' - \lambda V, \qquad V'(-c/2) = V'(c/2) = 0,$$

with

$$T_0 = \frac{V'' + \lambda V}{V'} = \frac{T_1 v_1' + T_2 v_2'}{v_1' + v_2'}.$$

Now, we have also $V(-c/2) = -V(c/2)$.

We claim that

$$T_0' \geqslant R + \frac{T_0^2}{n-1}. \tag{34}$$

This is enough to get the result. Indeed, the inequality (34) shows that the operator $L_0(v) = v'' - T_0 v'$ satisfies the $CD(R, n)$ hypothesis, and we may then apply the eigenvalue comparison result of Section 5 in the symmetric case (Corollary 6), and get

$$\lambda(R, n, \delta, -\delta/2) = \lambda \leqslant \lambda(R, n, 2c, -c/2),$$

which in turn gives $2c = \delta_0(a) \geqslant \delta$.

It remains to prove the claim, which is a consequence of Proposition 2 of Section 4.

First, set $f_i = \lambda \frac{v_i}{v_i'}$, and introduce

$$S = T_1 + T_2; \qquad D = T_1 - T_2; \qquad s = f_1 + f_2; \qquad h = f_1 - f_2.$$

Using the differential equation satisfied by T_1 and T_2, together with the differential equation satisfied by v_1 and v_2, using moreover the fact that v_1' and v_2' are positive on $(-d/2, d/2)$, a simple computation shows that

$$\text{Ineq } 34 \Leftrightarrow D\left[h+\frac{n}{n-1}D\right]\geqslant 0.$$

A simple inspection of the different cases shows that D has always the same sign as T' (remember that T' keeps a constant sign in any of the intervals we consider here).

Therefore, we just have to prove that

$$h+\frac{n}{n-1}D\geqslant 0, \qquad \text{when} \quad T'>0,$$

and that

$$h+\frac{n}{n-1}D\leqslant 0, \qquad \text{when} \quad T'<0.$$

Now, if we recall the differential equation satisfied by f_i

$$f_i'=\lambda-T_if_i+f_i^2,$$

we get immediately that

$$h'=sD/2-hS/2+sh.$$

Assume that we work in the situation where $T'>0$. Inequality (2) tells us that

$$f_i'>\frac{n}{n-1}T_i',$$

and therefore that

$$s'>\frac{n}{n-1}S'.$$

Since by construction $s(0)=S(0)=0$, we get from the previous

$$s<\frac{n}{n-1}S \qquad \text{on } (0, c].$$

Now, consider the function

$$F = h + \frac{n}{n-1} D.$$

Remark that both h and D are symmetric functions on $[-c, c]$, and therefore we just have to look at F the interval $[0, c]$.

An asymptotic expansion easily shows that at the boundary c, h goes to $-D(c)/2$, and therefore the inequality is true at the boundary.

At any point x where $F(x) = 0$, a simple computation shows that

$$F' = \left(s - \frac{1}{2} S\right) F + \frac{n+1}{2(n-1)} \left(\frac{n}{n-1} S - s\right) D.$$

Therefore, if we consider for example the case where $T' > 0$, then $F'(x) < 0$ at the last passage at 0, while $F(c) < 0$: this gives a contradiction.

The same argument holds true when $T' < 0$, since then $D < 0$ on the interval.

Thus we completed the proof.

8. COMPARISON THEOREM FOR SPECTRAL GAP

We may now collect all the different results of the previous sections to give the main theorem of this paper

THEOREM 14. *Let M be a compact Riemannian manifold without boundary or with a convex boundary, and let L be an elliptic differential operator $L = \Delta + B$ with smooth coefficients. Suppose that L satisfies the curvature-dimension type inequality $CD(R, n)$, ($R \in \mathbb{R}$, $n \in (1, \infty]$), and that the diameter is bounded above by d. Let λ_1 be a nonzero real eigenvalue (with Neumann boundary condition if the boundary is not empty). Then $\lambda_1 \geqslant \hat{\lambda}(R, n, d)$, where $\hat{\lambda}(R, n, d)$ denotes the first nonzero eigenvalue of the problem*

$$v'' - Tv' = -\lambda v, \qquad \text{on } \left(-\frac{d}{2}, \frac{d}{2}\right),$$

$$v'\left(-\frac{d}{2}\right) = 0, \qquad v'\left(\frac{d}{2}\right) = 0,$$

where the function T is

$$T=\sqrt{(n-1)\,R}\,\tan\left(\sqrt{\frac{R}{n-1}}\,x\right) \qquad \text{if} \quad R>0 \text{ and } n<\infty \tag{35}$$

$$T=-\sqrt{-(n-1)\,R}\,\tanh\left(\sqrt{\frac{-R}{n-1}}\,x\right) \qquad \text{if} \quad R<0 \text{ and } n<\infty \tag{36}$$

$$T=0 \qquad \text{if} \quad R=0 \text{ and } n<\infty \tag{37}$$

$$T=Rx \qquad \text{if} \quad n=\infty \tag{38}$$

This result applies in particular to symmetric operators $L=\Delta+\nabla h$, in which case λ_1 is the best constant λ in the Poincaré inequality

$$\lambda\left[\int f^2\,d\mu-\left(\int f\,d\mu\right)^2\right]\leqslant\int|\nabla f|^2\,d\mu,$$

where μ is the reversible measure $\mu(dx)=\exp(h(x))\,m(dx)$, m is the Riemannian measure.

In the case where $L=\Delta$, this applies in particular if n is the dimension of the manifold M and R is a lower bound of the Ricci tensor on M.

Proof. We just follow the proof of the easier result Corollary 6 of Section 5. Let u be an eigenvector with eigenvalue λ_1. We may assume that u is such that $\min(u)=-1$. Then, let $m_u=\max(u)$. Then we know that $m_u>0$. We may as well assume that $0<m_u\leqslant 1$, since if it is not the case, we may replace u by $-u/m_u$.

Now, let $L_{R,n}$ be the corresponding one dimensional operator satisfying $CD(R,n)$, as listed in the theorem. Then, we may use the results (9), (11), and (10) of Section 6, together with the results (1) and (2) of Section 3: there exists an interval $[a,b]$ which has λ_1 as eigenvalue for $L_{R,n}$ with Neumann boundary conditions, and, such that the corresponding eigenvector v of the model satisfies $v(a)=-1$, and $v(b)=m_u$.

Therefore, we may apply the main comparison result (8) of Section 6: the function $g=v^{-1}\circ u$ is such that $|\nabla g|\leqslant 1$. If p_1 and p_2 are two points on M such that $u(p_1)=-1$ and $u(p_2)=m_u$, then $g(p_1)=a$ and $g(p_2)=b$, and, by the definition of the distance function

$$d\geqslant d(p_1,p_2)\geqslant g(p_2)-g(p_1)=b-a.$$

Now, we may use Theorem 13 of Section 7:

$$\lambda_1 = \lambda(R, n, b-a, a) \geqslant \lambda(R, n, b-a, -(b-a)/2) = \hat{\lambda}(R, n, b-a).$$

Since the function $\hat{\lambda}(R, n, d)$ decreases with d, we get the result.

ACKNOWLEDGMENTS

The authors acknowledge the support through the TMR Contract GRBFMRXCT960075 while the second author was a member of the Department of Mathematics, Imperial College.

REFERENCES

1. D. Bakry, On Sobolev and logarithmic Sobolev inequalities for Markov semigroups, *in* "New Trends in Stochastic Analysis" (K. D. Elworthy, Ed.), pp. 43–75, World Scientific, Singapore, 1997.
2. D. Bakry and M. Emery, Diffusions hypercontractives, *in* "Sém. de Prob. XIX," Lecture Notes in Math., Vol. 1123, pp. 177–206, Springer-Verlag, Berlin, (1985).
3. D. Bakry and M. Ledoux, Sobolev inequalities and Myers's diameter theorem for an abstract Markov generator, *Duke Math. J.* **85**(1) (1996), 253–270.
4. D. Bakry and Z. Qian, On Harnack estimates for positive solutions of the heat equation on a complete manifold, *C. R. Acad. Sci. Paris* **324**(1) (1997), 1037–1042.
5. D. Bakry and Z. Qian, Harnack inequalities on a manifold with positive or negative Ricci curvature, *Rev. Math. Iberoamericana* **15**, No. 1 (1999), 143–179.
6. P. H. Bérard, "Spectral Geometry: Direct and Inverse Problems," Lecture Notes in Mathematics, Vol. 1207, Springer-Verlag, Berlin, 1986.
7. M. Berger, P. Gauduchon, and M. Mazet, "Le Spectre d'une Variété Riemanniene," Lecture Notes in Mathematics, Vol. 194, Springer-Verlag, Berlin, 1974.
8. S. Bochner, Vector fields and Ricci curvature, *Bull. Amer. Math. Soc.* **52**(1980), 776–797.
9. P. Buser, On Cheeger's inequality $\lambda_1 \geqslant h^2/4$, *in* "Geometry of the Laplace Operator," Proc. Symp. Pure Math., Vol. 36, pp. 29–77, Am. Math. Soc., Providence, 1980.
10. P. Buser, A note on the isoperimetric constant, *Ann. Sci. E'cole Norm. Sup.* **15**(4) (1982), 213–230.
11. P. Buser, "Geometry and Spectra of Compact Riemann Surfaces," Birkhäuser, Basel 1992.
12. R. Brooks, "Spectral Geometry," Cambridge Univ. Press, Cambridge, UK.
13. K. R. Cai, Estimate on lower bound of the first eigenvalue of a compact Riemannian manifold, *Chin. Ann. Math., Ser. B* **12**(3) (1991), 267–271.
14. I. Chavel, "Eigenvalues in Riemannian Geometry," Academic Press, New York, 1984.
15. I. Chavel, "Riemannian Geometry: A Modern Introduction," Cambridge Tracts in Math., Vol. 108, Cambridge Univ. Press, Cambridge, UK, 1993.
16. J. Cheeger, "A Lower Bound for the Smallest Eigenvalue of the Laplacian, Problems in Analysis," Symposium in Honor of S. Bochner, pp. 195–199, Princeton Univ. Press, Princeton, NJ, 1970.
17. J. Cheeger, G. Gromov, and M. Taylor, Finite propagation speed, kernel estimates for functions of Laplace operator, and the geometry of complete Riemannian manifolds, *J. Diff. Geom.* **17** (1982), 15–53.

18. J. Cheeger and D. Ebin, “Comparison Theorems in Riemannian Geometry,” North Holland, Amsterdam, 1975.
19. M. F. Chen and F. Y. Wang, General formula for lower bound of the first eigenvalue on Riemannian manifolds, *Sci. China (Ser. A)* **40**(4) (1997), 384–394.
20. R. Chen, Eigenvalue estimate on a compact Riemannian manifold, *Amer. J. Math.* **111** (1989), 769–781.
21. R. Chen, Neumann eigenvalue estimate on a compact Riemannian manifold, *Proc. Amer. Math. Soc.* **108** (1990), 961–970.
22. R. Chen and P. Li, On Poincaré type inequalities, *Trans. Amer. Math. Soc.* **349**(4) (1997), 1561–1585.
23. S. Y. Cheng, Eigenvalue comparison theorems and its geometric applications, *Math. Zeit.* **143** (1975), 289–297.
24. S. Y. Cheng and P. Li, Heat kernel estimates and lower bounds of eigenvalues, *Comment. Math. Helv.* **56** (1981), 327–338.
25. S. Y. Cheng and S.-T. Yau, Differential equations on Riemannian manifolds and their geometric applications, *Comm. Pure Appl. Math.* **28** (1975), 333–354.
26. B. Croke, Some isoperimetric inequalities and eigenvalue estimates, *Ann. Sci. Éc. Norm. Sup. 4e série* **13** (1980), 419–435.
27. B. Davies, “Heat Kernel and Spectral Theory,” Cambridge Tracts in Math. 92, Cambridge Univ. Press, Cambridge, UK, 1990.
28. A. Debiard, B. Gaveau, and E. Mazet, Théorèmes de comparaison en géométrie Riemannienne, *Publ. RIMS, Kyoto Univ.* **12** (1976), 391–425.
29. F. R. K. Chung, A. Grigor’yan, and S.-T. Yau, Upper bounds for eigenvalues of the discrete and continuous Laplace operators, *Adv. Math.* **117** (1996), 165–178.
30. S. Gallot, Inégalités isopérimétriques et analytiques sur les variétés riemanniennes, *Astérique* **163–164** (1988), 31–91.
31. D. Gilbarg and N. S. Trudinger, “Elliptic Partial Differential Equations of Second Order,” Springer-Verlag, Berlin, 1977.
32. M. Gromov, Paul-Levy’s isoperimetric inequality, 1980.
33. F. Jia, Estimate of the first eigenvalue on a compact Riemannian manifold with negative lower bound of Ricci curvature, *Chin. Ann. Math.* **12**(4) (1991), 496. [In Chinese]
34. P. Kröger, On the spectral gap for compact manifolds, *J. Diff. Geom.* **36** (1992), 315–330.
35. P. Li, A lower bound for the first eigenvalue of the Laplacian on a compact manifold, *Indiana U. Math. J.* **28** (1979), 1013–1019.
36. P. Li and S.-T. Yau, Eigenvalues of a compact Riemannian manifold, AMS Symposium on Geometry of the Laplace Operator, XXXVI, Hawaii, pp. 205–240, 1979.
37. P. Li and S.-T. Yau, On the parabolic kernel of the Schrödinger operator, *Acta Math.* **156** (1986), 153–201.
38. A. Lichnerowicz, “Géometries des Groupes des Transformations,” Paris, Dunod, 1958.
39. J. Milnor, “Morse Theory,” Ann. Math. Studies, Vol. 51, Princeton Univ. Press, Princeton, NJ, 1963.
40. S. B. Myers, Connections between differential geometry and topology, *Duke Math. J.* **1** (1935), 376–391.
41. M. Obata, Certain conditions for a Riemannian manifold to be a sphere, *J. Math. Soc. Japan* **14** (1962), 333–340.
42. W. T. Reid, “Ordinary Differential Equations,” Wiley, New York, 1971.
43. R. Schoen and S.-T. Yau, “Lectures on Differential Geometry,” International Press, 1994.
44. R. Schoen, S. Wolpert, and S. T. Yau, Geometric bounds on the low eigenvalues of a compact surface, *in* “Geometry of the Laplace Operator,” Proc. of Sym. in Pure Math. Vol. 36, 279–285, 1980.

45. I. M. Singer, B. Wong, S. S. T. Yau, and S. T. Yau, An estimate of the gap of the first two eigenvalues of the Schrödinger operator, *Ann. Scuola Norm. Pisa* **12** (1985), 319–333.
46. C. A. Swanson, "Comparison and Oscillation Theory of Linear Equations," Academic Press, New York, 1968.
47. Y. Xu, The first nonzero eigenvalue of Neumann problem on Riemannian manifolds, *J. Geometric Anal.* **5**(1) (1995), 151–165.
48. H. C. Yang, Estimate of the first eigenvalue on a compact Riemannian manifold with negative lower bound of Ricci curvature, *Sci. China (Ser. A)* **32**(7) (1984), 689.
49. S.-T. Yau, Isoperimetric constants and the first eigenvalue of a compact manifold, *Ann. Sci. Éc. Norm. Sup.* **8** (1975), 487–507.
50. S.-T. Yau, Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry, *Indiana Math. J.* **25** (1976), 659–670.
51. S.-T. Yau, Nonlinear analysis in geometry, *Enseign. Math.* **33**(2) (1987), 109–158.
52. J. Q. Zhong and H. C. Yang, On the estimate of the first eigenvalue of a compact Riemannian manifold, *Sci. Sinica Ser. A* **27**(12) (1984), 1265–1273.